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A PIECEWISE-LINEAR SWITCHING FUNCTION FOR QUASI-MINIMUM-TIME CONTROL

by

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ABSTRACT

The design of controllers to provide quasi-minimum time-response is investigated. With the control law to be realized as a piecewise-linear combination of the instantaneous state values, the problem is to select the parameters which define the best suboptimal switching surface.

A worst-case response-time performance index is defined and shown to be suitable for use in the suboptimal controller design. A proposed design technique is described and used to determine controllers for second-order systems. The results of the illustrative examples are compared with controllers designed using a least squares polynomial fitting technique. The comparison shows that the proposed method designs significantly better controllers in terms of deviation from optimal response time.

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

1. INTRODUCTION

This thesis presents a proposed method for the design of controllers that provide quasi-minimum-time response of systems described by the set of state equations

$$(1.1) \qquad \dot{\underline{x}}(t) = A\underline{x}(t) + \underline{B}u(t)$$

where

$\underline{x}$ is an nxl state vector

A is the nxn plant matrix

u is the bounded scalar control

$\underline{B}$ is the nxl distribution matrix of the control.

In order to insure that an optimal control exists, the plant transfer function is restricted to one having only real, negative eigenvalues, and it is further specified that it have no zeros. Additionally, all states must be observable and the control effort bounded.

For plants of the type described, previous studies [1,3] have shown that a unique optimal control will exist that transfers the plant from any point in the state space to the origin with no more than n-l switches of the control. Furthermore, the control will always be at its maximum positive or negative value (i.e. the bang-bang control). Since the control is always at its maximum value, an ideal relay can be used to provide the control to the plant if some other device can be constructed to properly control the relay as shown in figure 1.

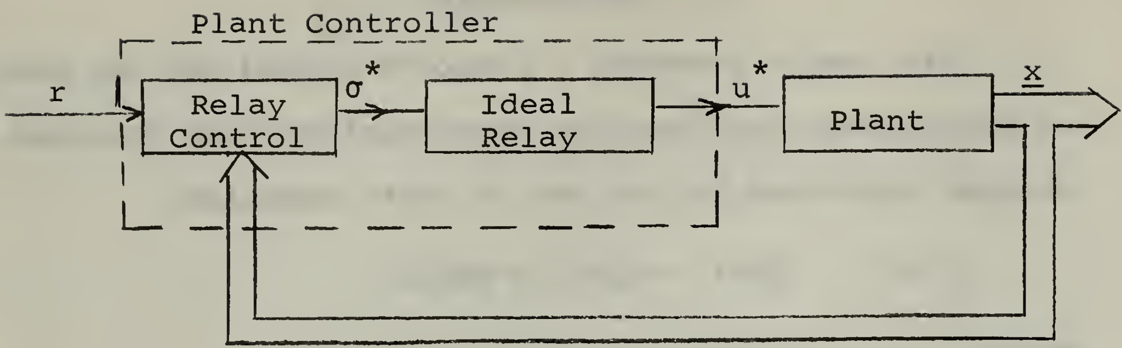


Figure 1. Optimal Controller

From the figure it is seen that

$$(1.2) \qquad u = \operatorname{sgn} \sigma$$

Therefore, the controller must generate the proper σ to transfer the plant from $\underline{x}(0)$, the initial condition vector, to $\underline{x}(f)$, the desired final condition vector, in minimum time. When the plant input, r, is zero the problem becomes that of the linear regulator since the desired state of the system at the final time is the state space origin. If r is non-zero but restricted to an allowable class of input functions, the effect may be described as a shift in the state origin.

Time-optimal control can be implemented for the systems considered in this study by one of two general methods. In the first method, a digital computer is used as the controller. The computer must rapidly calculate the control using the differential equations of the plant and the specific initial condition. The disadvantages of this

method are that it is open-loop which is undesirable since unexpected disturbances are not taken into account, and it is possible that the computer may not be able to generate the solution rapidly enough to allow proper switching of the control. The second method of generating the optimal control is to determine σ as a function of the state variables of the plant with some electronic device. While more appealing than the first method, this method is limited in application by the fact that a closed form expression for the switching function is not usually obtainable for systems of higher than second order.

In many instances the complexity and cost of the optimal controllers described above is not justified from an engineering point of view. That is, a less complex and more easily implemented controller may be acceptable if it can be designed to deviate from optimal performance by only a "small" amount. The problem then becomes that of developing a design procedure which results in an easily realized controller that produces acceptable, suboptimal response of the plant.

In this investigation a suboptimal design method is developed using a controller of a pre-specified form. The parameters of the controller are adjusted to provide acceptable time response of the plant. In the state space the optimal switching function, σ^*, consists of a hypersurface that divides the space into regions of +M and -M control, where $|M|$ is the absolute magnitude of the bound-

ed control. Throughout the remainder of this study it will be assumed that all equations are normalized so that $|M| = 1$. It would seem that the switching function generated by the suboptimal controller should be a hypersurface that lies "close" to that generated by the optimal controller.

The suboptimal switching function used in this work is a piecewise-linear combination of the state variables of the system.

In Section 2 the optimal and suboptimal switching surfaces are described. Selection of an index of performance is the topic in Section 3. An outline of a proposed design procedure for suboptimal controllers is presented in Section 4 and Section 5 gives the results of suboptimal controller designs using the proposed procedure and compares the designs with controllers designed by another method.

2. THE PIECEWISE-LINEAR SWITCHING FUNCTION

2.1 The Optimal Switching Surface

For time-optimal control the switching function divides the state space into areas of ± 1 control. The closed form of the optimal switching function as a function of the system state variables can be expressed as

$$\sigma^* = f^*(\underline{x}) = 0 \tag{2.1}$$

Alternatively, this function can be expressed as

$$\sigma^* = f^*(\underline{x}') - x_i = 0 \tag{2.2}$$

where

* indicates optimal

$\underline{x}'$ is the previous $\underline{x}$ vector with the x_i component removed

if the function is single valued in x_i. The function $f^*(\underline{x}')$ is independent of x_i and gives the x_i coordinate of the optimal switching surface. For plants of the form prescribed for this study, Smith [2] has shown that the surface can be made single valued in the uncoupled state variables by transforming to Jordan canonical form. In many instances, however, the optimal switching surface is already single valued in at least one state variable so that no transformation is necessary.

Since n-1 switches of the control are required to transfer the system from any initial point to the state space origin in minimum time, the optimal switching surface may be generated by integrating the state equations

backwards in time and switching the control n-2 times. This procedure generates one half of the optimal surface. The other half is easily obtained since the surface is an odd function of the state variables for a linear system with symmetrical control.

The optimal switching surface and trajectories for a second order, double integrator plant may be obtained by solving the differential equations of the plant,

$$\dot{x}_1(t) = x_2(t) \qquad (2.3)$$

$$\dot{x}_2(t) = u$$

where

$u = \pm 1.$

Using classical methods to solve these equations with a given set of initial conditions $\underline{x(t_o)}$ yields

for $u = -1$

$$x_1(t_f) = x_1(t_o) + tx_2(t_o) - \frac{1}{2} t \qquad (2.4a)$$

$$x_2(t_f) = x_2(t_o) - t$$

for $u = +1$

$$x_1(t_f) = x_1(t_o) + tx_2(t_o) + \frac{1}{2} t \qquad (2.4b)$$

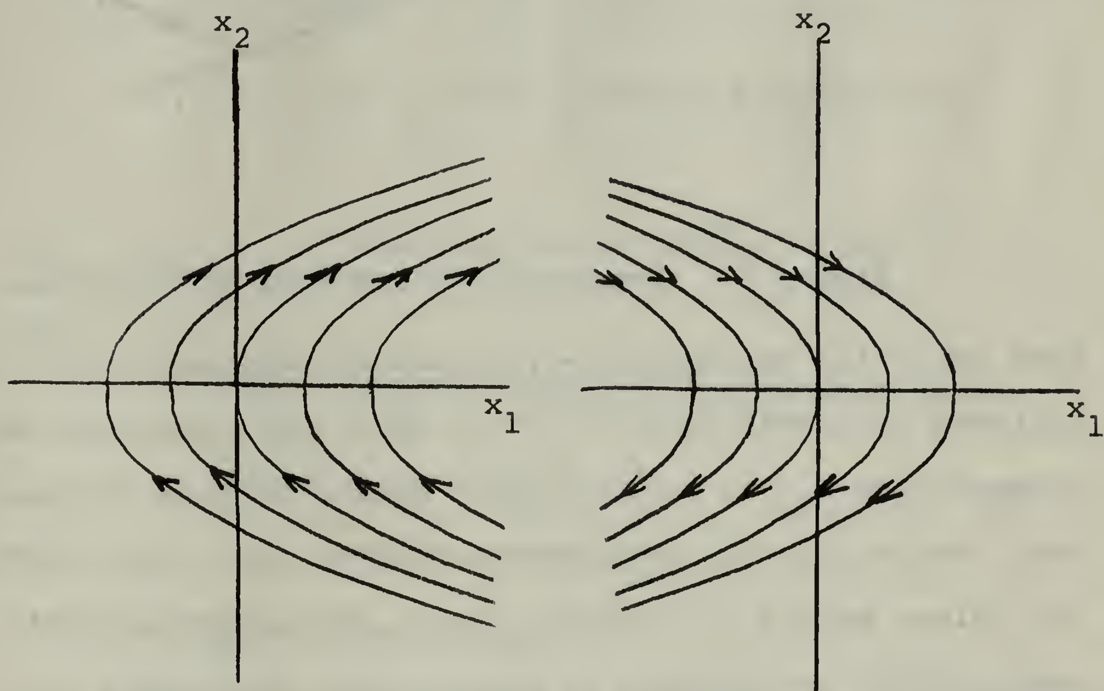
$$x_2(t_f) = x_2(t_o) + t.$$

The minimum-time trajectories in the state space may be obtained by eliminating the variable t and solving for

$x_1(t)$ in terms of $x_2(t)$ which results in the well known families of parabolas

$$
(2.5) \qquad
\begin{aligned}
x_1(t) &= \frac{1}{2}\, x_2(t) + C_1 \qquad \text{for } u = -1 \\
x_1(t) &= \frac{1}{2}\, x_2(t) + C_2 \qquad \text{for } u = +1
\end{aligned}
$$

The families are shown in figure 2.



(b) $u = +1$ (a) $u = -1$

Figure 2. Minimum-Time Trajectories

To force the states of the system from a set of random initial conditions in the state space to the origin in minimum time, the families shown above must, generally, be combined to produce the minimum-time trajectories as shown by Pontryagin [1]. Four typical trajectories are shown in figure 3.

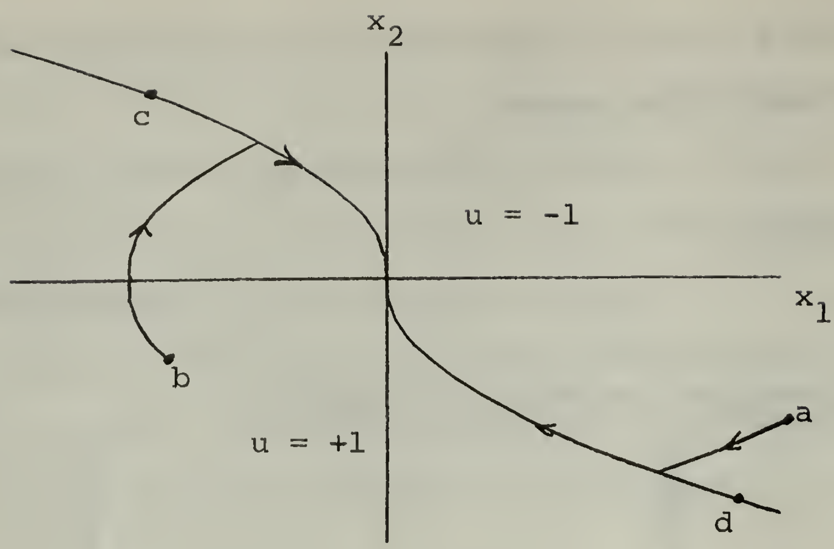


Figure 3. Composite Minimum-Time Trajectories

From point (a) the minimum-time control sequence is u = -1 followed by a switching to u = +1 when the trajectory intersects the u = +1 parabola that passes through the origin. For point (b) the opposite sequence must be followed. Since points (c) and (d) are on the parabolas that pass through the origin, no switches of control are required. (c) and (d) can reach the origin with n-2 switches of control, therefore they lie on the optimal switching surface, in this case a line, which from equation(2.4) is

$$(2.6) \qquad x_1 = -\frac{1}{2} x_2 \,|\, x_2 \,|$$

2.2 The Piecewise-Linear Switching Surface

Since the suboptimal switching function specified in this study is a piecewise-linear function of the state variables of the system it can be expressed as

$$(2.7) \qquad \hat{\sigma} = \hat{f}(\underline{x}') - x_i = 0$$

where

$\wedge$ indicates suboptimal

For the second-order systems considered in this investigation the switching surface reduces to a line in the state space and is given by

$$x_2 = [P(1)/P(2)]x_1 \, \mathbb{1}(x_1)+[-P(1)/P(2)x_1+P(1)] \, \mathbb{1}(x_1-P(2) + \sum_{i=2}^{L}\Big\{A(i)x_1 \, \mathbb{1}[x_1-P(2i-2)] + [-A(i)x_1+P(2i-1)] \, \mathbb{1}[x_1-P(2i)]\Big\} \qquad (2.8)$$

where

$$A(i) = [P(2i-1)-P(2i-3)]/[P(2i)-P(2i-2)]$$

$$\mathbb{1}(\text{arg}) = 1 \qquad (\text{arg}) \geq 0$$
$$= 0 \qquad \text{otherwise}$$

L = number of PWL segments

$P(\text{odd}) = x_2$ coordinates of breakpoints in the PWL surface

$P(\text{even}) = x_1$ coordinates of breakpoints in the PWL surface.

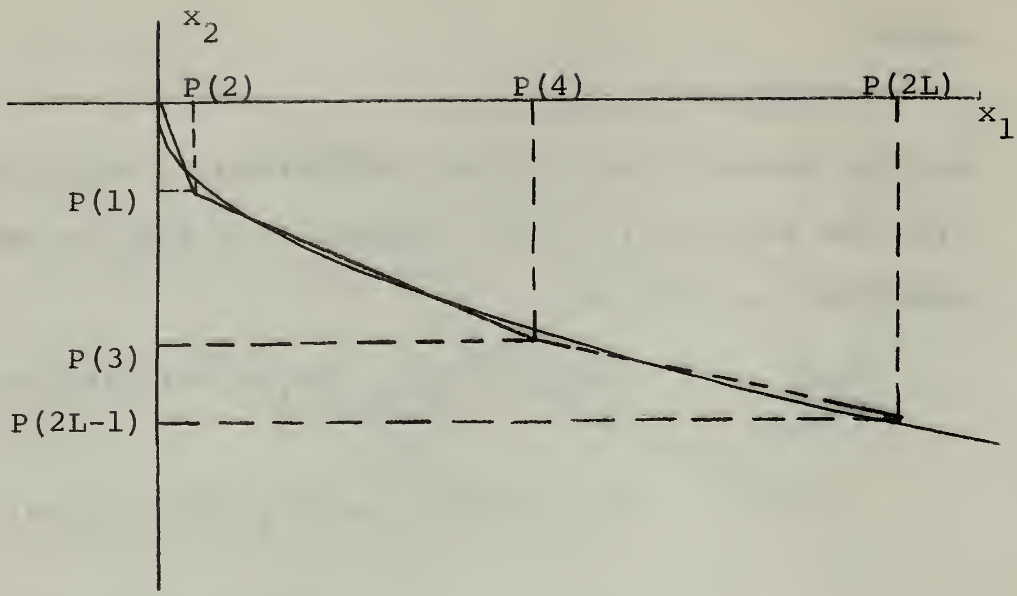


Figure 4. The PWL Switching Surface

For the purposes of this paper it will be assumed that the range of initial conditions is known. P(2L) will be determined by this expected range. The design procedure must determine the best, in some sense, values for the other 2L-1 parameters of the surface. The number of segments, L, in the pwl surface is not prespecified in this design procedure since it is desired to complete the controller using the minimum number of segments.

The implementation of the pwl switching surface is easily accomplished using resistors, diodes and batteries. Redderson [4] has shown construction of pwl function generators using these elements. The objective of this investigation is the development of a method for selecting the best parameters describing the pwl surface. The remainder of this study will deal with the design method develop-

ment -- not the physical construction of the controller.

2.3 Effects of PWL Switching

As described previously, a maximum of n-1 switches of control are necessary for minimum-time control. Since this study deals with suboptimal control, it is of interest to investigate the trajectories and switchings caused by the suboptimal switching surface. Consider the one segment pwl switching surface shown in figure 5.

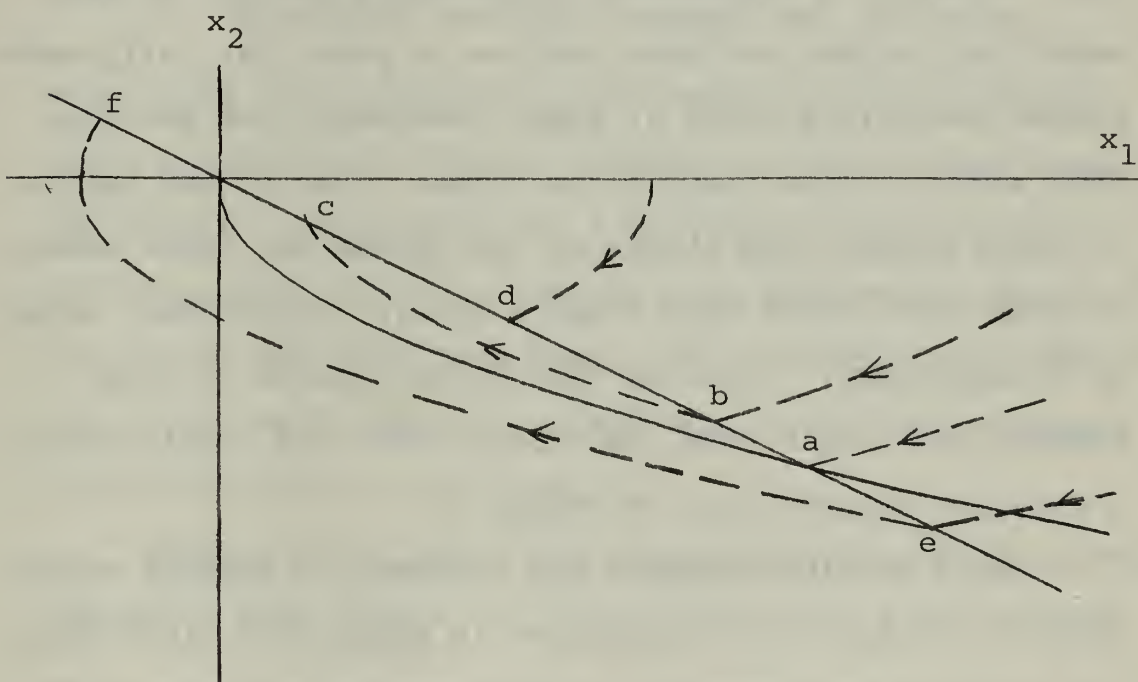


Figure 5. Effects of PWL Switching

For the case of one segment, the pwl switching problem reduces to the linear switching problem using tachometer feedback for a relay controller as described by Gibson [5] and others. In figure 5 the trajectory that intersects the pwl surface at point (a) will be switched optimally and will reach the origin in minimum time. All

other trajectories shown will deviate from the optimal and will, therefore, be suboptimal. The trajectory intersecting the pwl surface at point (b) will follow a u = +1 trajectory until reaching (c) at which time it will chatter or bump down toward the origin. Any trajectory intersecting between point (d), the point of tangency with an optimal trajectory, and the origin will immediately begin to chatter and will continue to do so until reaching the origin. Finally, a trajectory intersecting the pwl surface below the optimal surface, such as at point (e), will move around the origin until it again intersects the pwl surface and will then bump to the origin. Due to the nature of relay chatter the states of the system can never exactly reach the origin when bumping occurs in the final stage of a trajectory. A system will be considered to have reached the origin when the states enter and remain within a circular neighborhood or radius r.

For a multiple-segment pwl surface the effects noted above all occur, but the problem is compounded since they may occur on each segment. A two-segment pwl switching surface is shown in figure 6.

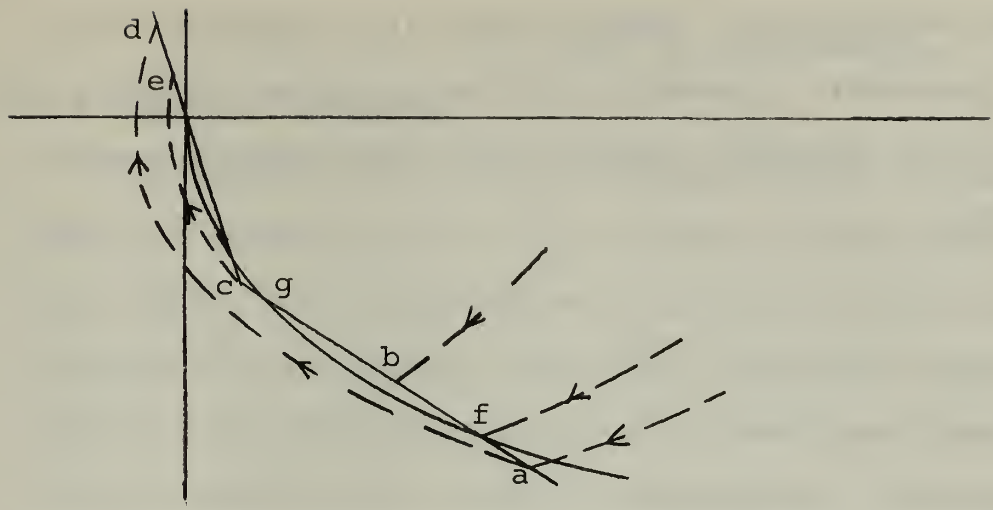


Figure 6. Multiple-Segment PWL Switching

From point (a) a trajectory will move around the origin until intersecting the pwl surface at (d) and then bump to the origin. From point (f) a trajectory will follow a $u = +1$ parabola to (g), bump to (c), move around the origin to (e) and then bump down. The trajectory intersecting at (b) will bump to (c) and then follow the sequence described above to the origin.

Redderson [4] gives the time it takes a second-order system to bump from $x_1(t_o)$ to $x_1(t_f)$ along a pwl segment as

$$(2.9) \qquad t_b = \frac{1}{b} \operatorname{Log}\left[\frac{a+bx_1(t_f)}{a+bx_1(t_o)}\right]$$

where

b = slope of the pwl segment

$a = x_2$ intercept of the segment.

To achieve good suboptimal performance this bump time must be minimized since it represents the major deviation from

optimal performance. Bumping must also be minimized because frequently repeated rapid changes of control are not desirable in physical systems due to the component wear produced.

3. SELECTION OF AN INDEX OF PERFORMANCE

3.1 General Comments

In order to evaluate any design method for the proposed suboptimal controller, an index of performance, or cost index, must be specified. The index should indicate how much the suboptimal controller causes the system to deviate from optimal performance. Since quasi-optimal control is desired, the optimal and suboptimal responses should deviate by only a small amount. The cost index selected should indicate close to optimal performance by having a small value which should increase as optimal and suboptimal control deviate to a greater extent. For time-optimal control the cost index is defined as

$$J = \int_{t_o}^{t_f} dt = t_f - t_o \tag{3.1}$$

Minimizing the above index leads to the time-optimal switching surface discussed in section 2.1.

The selected cost index should generate a smooth cost surface to allow easy location of the minimum. Three different types of cost indices may be defined for use in the suboptimal problem as described in this paper. The three types are

a) response-time summation indices

b) non-response-time indices

c) worst-case indices

These three types of indices will be discussed in the re-

mainder of this section.

3.2 Response-Time Summation Indices

Since the problem being studied is that of quasi-minimum-time control, a cost index that evaluates the design on the basis of the actual plant response times under pwl switching seems desirable. Many indices of this type may be defined. Some examples are given below.

(3.2)

1) $$J_1 = \frac{1}{K} \sum_{i=1}^{K} \hat{t}_i$$

2) $$J_2 = \frac{1}{K} \sum_{i=1}^{K} \hat{t}_i / t_i^*$$

3) $$J_3 = \frac{1}{K} \sum_{i=1}^{K} \hat{t}_i - t_i^*$$

4) $$J_4 = \frac{1}{K} \sum_{i=1}^{K} (\hat{t}_i - t_i^*)^2$$

5) $$J_5 = \frac{1}{K} \sum_{i=1}^{K} \left(\frac{\hat{t}_i - t_i^*}{t_i^*}\right)$$

6) $$J_6 = \frac{1}{K} \sum_{i=1}^{K} \left(\frac{\hat{t}_i - t_i^*}{t_i^*}\right)^2$$

where K = number of initial conditions

$\hat{t}_i$ = pwl response time for i^{th} initial condition

t_i^* = optimal response time for i^{th} initial condition

The indices defined above are summations of various measures of system response time and include effects from all initial conditions in the evaluation. A desirable feature for an index would be that it equally weight all initial conditions regardless of their location in the state space. Of the indices given above J_1 and J_3 give more weight to initial conditions further from the origin. The other indices give equal weight to all initial conditions since a normalizing factor, $1/t^*$, is used. It then appears that this normalizing factor should be used in the index finally selected for this study.

The indices defined above all possess a major limitation which appears only after a detailed study of the cost surfaces that they generate.

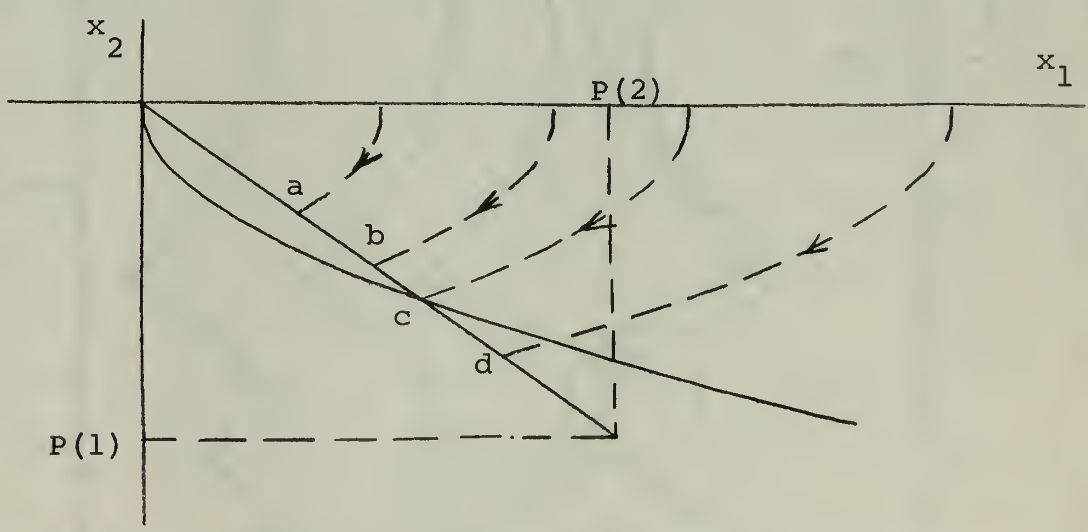
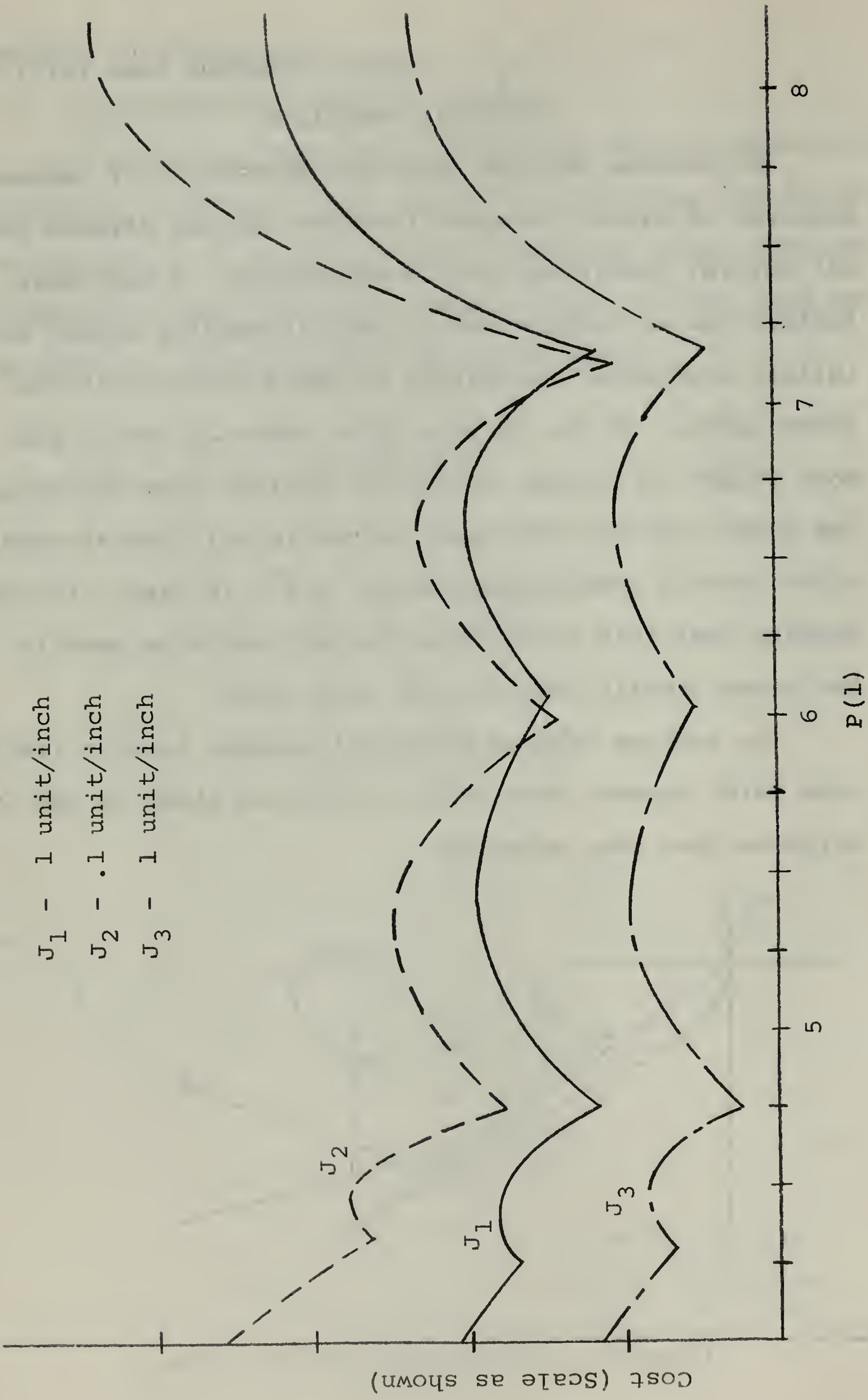


Figure 7. One-Segment PWL Switching



J_1 - 1 unit/inch
J_2 - .1 unit/inch
J_3 - 1 unit/inch
J_1
J_2
J_3
5
6
7
8
P(1)
Cost (Scale as shown)

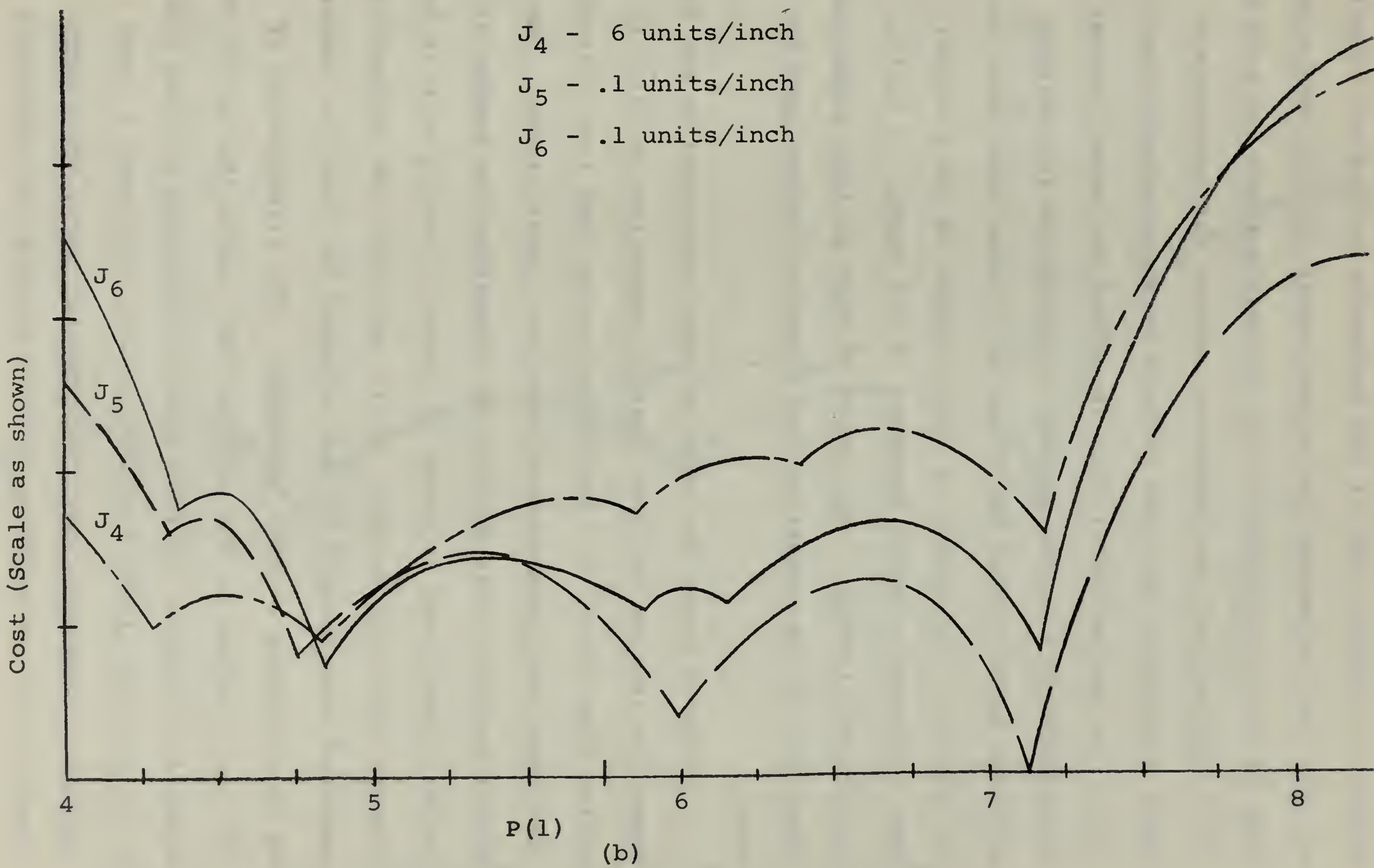


Figure 8. Cost Surfaces for Summation Indices

Figure 7 suggests that if the parameter values are adjusted so that the trajectory for a particular initial condition such as (c) is switched at the optimal surface, it will be truly optimal and the summation indices will generate a local minimum in the cost surface at this parameter setting. As the parameter P(1) is varied the other initial conditions will follow optimal trajectories for certain settings as described above and will cause additional local minima. The number of local minima in the cost surface increases with an increase in the number of initial conditions, however, the depth of each is less since each component in the summation has a smaller effect upon the total.

Figure 8 shows the cost surfaces generated by the six summation indices for four initial conditions as P(1) is varied from -4 to -8. The figure shows that four initial conditions may produce more than four local minima. The effect upon the cost surface of increasing the number of initial conditions is shown in figure 9 for J_3 as K is increased from four to ten. It is seen that the surface is smoothed considerably by the increase but that the number of local minima is also increased. The irregularity of the cost surface is the problem that limits the use of the summation indices. Normal search techniques are not able to locate the global minimum but will stop when a local minimum is reached. With the possibility of the search terminating upon reaching a local minimum the parameters

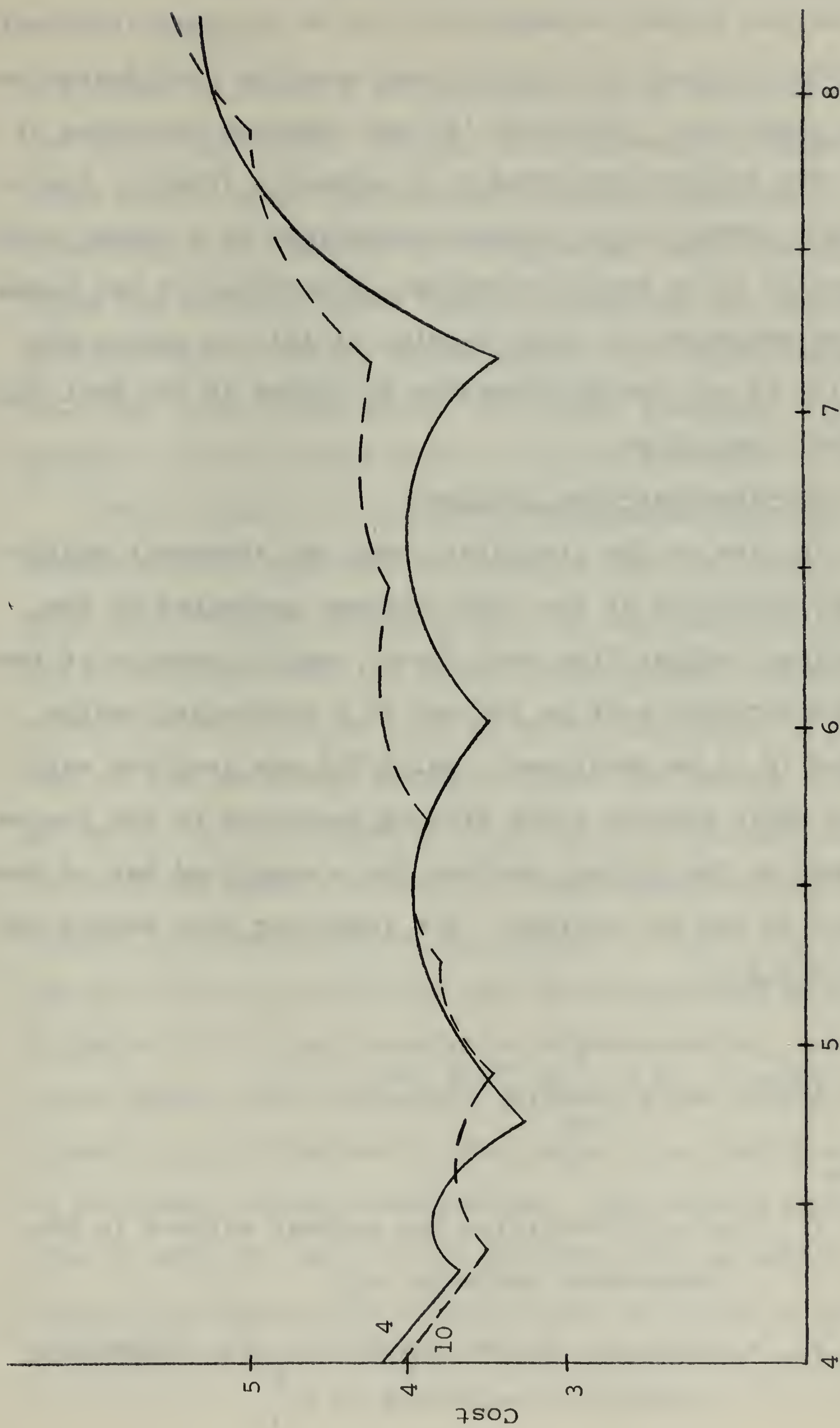


Figure 9. Cost Surfaces for 4 and 10 Initial Conditions

of the pwl surface selected may not be the best attainable and the designer will not be sure when the best design has been completed. Frederick [6] has improved the shape of the cost surface generated by a summation index by limiting the number of pwl surface parameters to a number which allows it to be easily searched. Limitation of the number of pwl parameters in this fashion is felt to reduce the ability of any design procedure to arrive at the best suboptimal controller.

3.3 Non-response-Time Indices

In view of the irregular shape and attendant searching difficulties of the cost surfaces generated by the summation indices discussed above, another measure of system performance must be defined if a controller design method is to be developed. Smith [2] has used the well-known least squares curve fitting technique to fit the pwl surface to the optimal surface for a specified set of breakpoints in the pwl surface. The index for this method may be defined as

$$(3.3) \qquad J = \sum_{J=1}^{P} (x_{ij}^{*} - \hat{x}_{ij})^2$$

where

x_{ij}^{*} = points describing the optimal surface in the independent variable, x_i

$\hat{x}_{ij}$ = points on the pwl surface in the independent variable corresponding to x_{ij}^{*}

P = number of points over which the pwl surface is fitted.

The object of Smith's design method is to minimize this index which represents the sum of the squared distances between points on pwl and optimal surfaces measured in the coordinate direction of the independent state variable. Using the analytic least squares technique the desired minimization is easily accomplished. For one pwl segment with one variable parameter this index generates a smooth parabolic cost surface which is easily searched.

While simple in concept and implementation, the above method suffers from the fact that it does not guarantee good time response of the plant. It only assures that the pwl surface will lie close to the optimal surface. Actual system response time is never considered in the design of the switching surface. After the design is completed it may be partially justified by using the controller with the plant for a set of representative initial conditions and observing the performance. This is an after the fact type of evaluation and the design must be completely redone if the performance is not acceptable. Also, as stated above, the breakpoints of Smith's pwl surface, the P(even) in the notation of this paper, are specified prior to the least squares minimization. This allows optimization of only one half of the available pwl parameters. It should be observed that if the design is found to be unacceptable it can be repeated with another set of break-

points. The problem with this procedure is that previous trials provide little information of value in selecting the breakpoints for future trials.

Redderson [4] also used least squares fitting techniques but defined his cost index as

$$(3.4) \qquad J = \sum_{i=1}^{P} \left(\delta_t^{\,2}\right)_i$$

where δ_t is defined as the transverse trajectory time between the pwl and optimal surfaces as shown in figure 10.

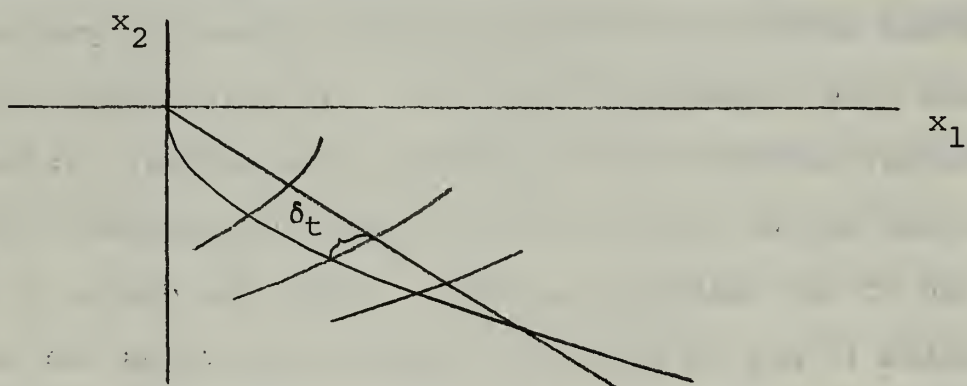


Figure 10. Transverse Trajectory Time

Minimizing this index relates the pwl fitting to system response time, but it is not really a measure of overall performance. Since it measures trajectory time between the surfaces it again, as does Smith's index, only assures that they will be close together. Actual system performance must be evaluated upon completion of the design using representative initial conditions. The advantage of this method over Smith's is that the breakpoints of the pwl surface, as well as the slopes, are allowed to vary which re-

sults in optimizing over all of the parameters. This should result in a better least squares fit and, if indeed there is a relationship between the goodness of fit and response time, better system performance. Since there is no definite relationship between these quantities additional response -time indices should be considered in an effort to provide better suboptimal response.

3.4 Worst-Case Response-Time Indices

The difficulties of local minima with the summation indices and of no real relationship between good least squares fit and good system response can be avoided by selecting an index that measures the worst deviation from optimal performance for the set of initial conditions. Three examples of this type of index are given below.

$$J_1 = \max_{\underline{x(0)}} \hat{t}_i / t_i^* \tag{3.5}$$

$$J_2 = \max_{\underline{x(0)}} (\hat{t}_i - t_i^*) \tag{3.6}$$

$$J_3 = \max_{\underline{x(0)}} (\hat{t}_i - t_i^*)/t_i^* \tag{3.7}$$

where

$\hat{t}_i$ = pwl response time for i^{th} initial condition

t_i^* = optimal response time for i^{th} initial condition

$\underline{x(0)}$ = initial condition vector.

The cost surface is generated by minimizing the indices with respect to the pwl parameters. The design can then be described as the min-max problem which has been studied by

von Neumann [7] and others.

$$(3.8) \qquad J^* = \min_{\underline{P}} \max_{x(0)} J$$

where

$\underline{P}$ is the vector of pwl parameters.

As noted by Kalman [8] the minimum-time problem can be regarded as a two-player zero-sum min-max game. That is, nature tries to maximize response time while man tries to minimize it.

Since it is desired that the index selected indicate desired performance and generate a smooth cost surface, the three indices were evaluated for the second-order one pwl segment case. Figure 11 shows the surfaces as P(1) varies from -7.45 to -50.7 with P(2) set at +16. J_1 and J_3 show smooth behavior over the range of the variable parameter. J_2 is also generally smooth but does show minor irregularities. The points plotted are the costs for a set of 24 initial conditions spaced along the x_1 coordinate axis.

If the min-max point, J^* as given by equation (3.8), can be found the designer is assured that system performance for all the initial conditions can be no worse than that indicated by J^*. The system performance for other initial conditions within the design range will be discussed in sections 4 and 5. Figure 11 shows that all three indices generate easily searched cost surfaces. For the design method to be presented in the next section the

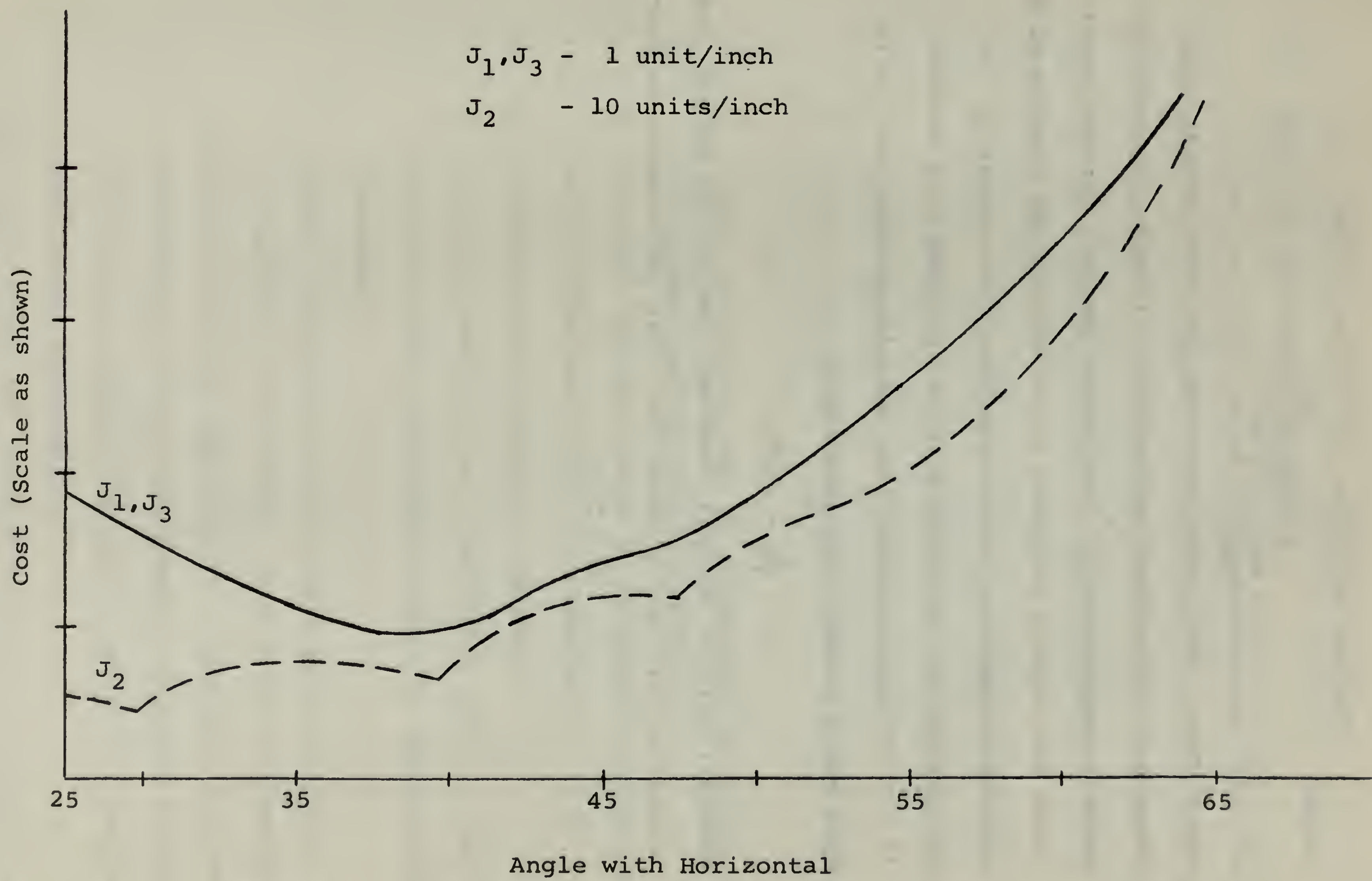


Figure 11. Cost Surfaces for Worst-Case Indices

index

$$(3.7) \qquad J = \max_{\underline{x(0)}} \left(\frac{\hat{t}_i - t_i^*}{t_i^*} \right)$$

was chosen. This index measures the worst deviation between pwl and optimal performance. It also includes the normalizing factor $1/t^*$ to insure that all initial conditions are equally weighted while looking for the one with maximum deviation. As an example of the necessity for this weighting, consider a deviation time of 2 seconds. If the optimal response time were 15 seconds this deviation would probably be acceptable.

$$J = \frac{2}{15} = .1333$$

However, with an optimal response time of one second, the normalization clearly shows that this performance is poor.

$$J = \frac{2}{1} = 2.0$$

4. DESIGN OF THE PWL SWITCHING SURFACE

4.1 Steps of the Design Procedure

In order to obtain acceptable performance from the pwl controller it is necessary to determine the parameters that minimize the cost index selected in the last section. The steps of the procedure to obtain these parameter values are listed below.

a. Select the region in the state space for which the design is desired and a set of initial conditions representative of this region.

b. Obtain the optimal response times for the selected initial conditions.

c. Select an initial set of parameters to define a one segment pwl switching surface.

d. Evaluate the cost index to determine if performance of the controller is acceptable. If it is acceptable the design is complete.

e. If performance is not acceptable, vary the parameters in some systematic manner to improve performance.

f. Repeat steps (d) and (e) until performance is acceptable or no further improvement is possible.

g. When no further improvement is possible and performance is still unacceptable, increase the number of pwl segments by one by defining a new set of initial parameters.

h. Repeat steps (d) through (g) until acceptable per-

formance results or no further improvement is possible.

Thus far, it has been stated that the design process is completed when a set of pwl parameters is found that produce "acceptable" performance of the plant. At this point it is necessary to define this acceptability criterion. The procedure outlined above, when used with the cost index selected in the last section, results in minimizing the normalized or percentage deviation time for the set of initial conditions. By defining the acceptability criterion as some percentage deviation from optimal response time,

$$(4.1) \qquad J_a = \frac{\Delta t_a}{t^*} \qquad \Delta t_a = (\hat{t}-t^*) \text{ acceptable}$$

the procedure assures that the system will respond at least this well, and for most initial conditions it will be better. It also completes the design using the minimum number of pwl segments and upon completion of each cycle indicates the improvement gained by the addition of the last segment.

The value selected for J_a is a function of many variables. The system being controlled, its employment, allowable cost and complexity of the controller are all factors that must enter into selection of the acceptability criterion. The designer must determine how close to the optimal he desires the pwl controller to operate in view of the above factors. After the pwl parameters are selected by the design procedure, the controller must be implemented

using pwl function generators to produce the control as a combination of the state variables as described in section 2.

4.2 Selection of the Initial Conditions and Their Range

While it would be desirable for the pwl controller to provide acceptable control for initial conditions throughout the entire state space, the proposed design method minimizes the cost index for the set of selected conditions. From an engineering standpoint this is not a serious restriction since for most physical systems a range of expected initial conditions can usually be predicted. The controller must, then, provide acceptable control over this range and not the entire state space. Consider a shipboard missile launcher as an example. Under normal conditions this system would be at some initial position with no velocity or acceleration. Assume that the system is modeled as a third order system with the states defined as

x_1 = angle of launcher with respect to some reference

x_2 = angular velocity

x_3 = angular acceleration.

In the three dimensional state space the possible initial conditions would be located along the x_1 coordinate axis. The range of the initial conditions is also limited in this example by the maximum train angle of the launcher which is usually restricted to somewhat less than 2π radians by such

things as electrical cable twist and structural interference.

It can be shown that initial conditions lying along the pwl surface generally respond more poorly than those located in other regions of the state space. In view of the engineering considerations mentioned above and the fact that placing the initial conditions along the pwl surface requires recomputation of optimal response times after each parameter perturbation, it was decided to place them along the positive x_1 axis for the second order examples presented in the next section. This corresponds to a system with initial position but no initial velocity. After the set of initial conditions is selected for which the controller is to be designed, the optimal response times necessary for normalization can be generated once and need not be recomputed after each change in parameter settings. Selecting the initial conditions in this manner introduces an additive constant into the response times, because each trajectory moves across the state space in an optimal manner until it reaches the pwl or optimal switching surface at which time it begins to deviate from optimal response.

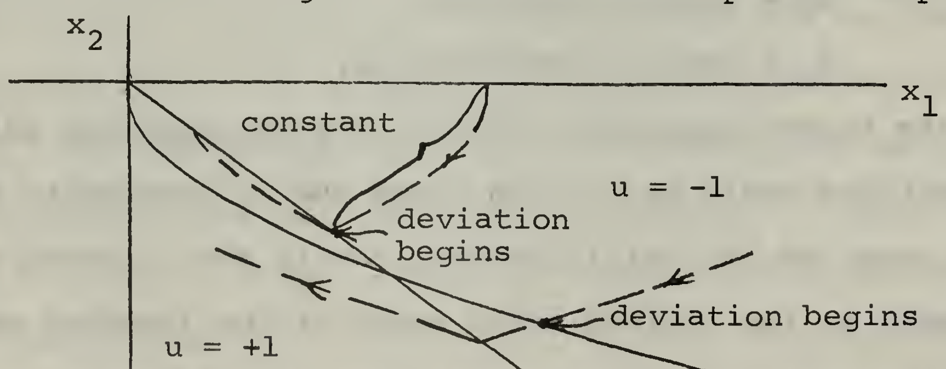


Figure 12. Additive Constant in the Response Time.

Figure 12 shows the additive constant that results from the trajectories crossing the state space before starting to deviate. Notice that the normalized deviation time index used in this design procedure removes this constant since it measures only the deviation from optimal response. If a summation of response times index had been selected this constant could affect the optimization process since it might reduce the differences between various controllers so that the best design was not readily apparent.

Since the minimization of the cost index is performed over a finite set of initial conditions, it is necessary, upon completion of the design, to evaluate the system response for other initial conditions in the range of interest. This check is required to insure that there are no points in the range that result in unacceptable performance. The smooth shape of the cost surface should insure that once the best parameter settings are found for a large enough set of initial conditions, all others in the range will respond acceptably also.

4.3 Selection of the Number of PWL Segments

In the design of the pwl controller, it is desirable to generate the control using the minimum number of pwl segments. This produces both the least complex and least costly solution to the problem while still providing acceptable performance. The design method outlined in section 4.1 has this desirable feature. If the performance with L segments is acceptable, the design is completed and there

is no need to evaluate L+1 segments. Smith [2] starts his design by specifying the number of pwl segments to be used and then attempting to provide good performance by adjusting the segment slopes. Redderson's [4] method is similar in that the number of segments is prespecified, but both slopes and lengths of the segments are allowed to vary.

If an acceptability criterion that allows appreciable deviation from optimal performance is selected, linear switching may be adequate. Since the proposed design method starts with one segment, it will be evident if this is true. If a specified number of segments is determined to provide unacceptable control, only one more segment is added and the design is terminated when acceptability is reached. Therefore, use of the minimum number of segments is insured.

Additionally, it is possible to monitor the approach to acceptability by comparing the best cost obtained before and after adding a segment. If the improvement is small, the acceptability criterion should be reviewed to determine if it is too strict before adding additional segments which increase cost and complexity. At all stages in the design procedure it is well to monitor the effect of adding a segment to insure that the design is completed using the minimum number of segments.

4.4 Searching the Cost Surface

With the cost index, the set of initial conditions and the initial pwl segments selected, once the method of

searching the cost surface is described the design method is complete. For a second-order pwl switching surface of L segments the cost surface is a function of 2L-1 variables. This 2L-1 dimensional space must be searched to determine the best parameter settings for the controller. In view of the high dimensionality of the cost surface, an efficient search technique must be used to minimize the computer time required for the design process.

Hooke and Jeeves [9] have developed a direct search technique and shown that computation time increases roughly as the first power of the number of variables not as the cube as is the case for most classical minimization techniques. This direct or pattern search makes use of past successful parameter adjustments to predict possible good future moves. The search establishes a pattern of improvement based upon the success of individual parameter perturbations. The size of the pattern grows with continued success, thus accelerating the search toward the minimum. If a move does not result in improvement, the size of the perturbation is reduced until improvement results or a limiting criterion is reached and the search is terminated. A more detailed description of the pattern search may be found in Appendix A.

The pattern search was chosen for the minimization of the cost index in this study because of its relative simplicity and efficiency. It is noted that this search is dependent upon the perturbation step size and the initial

parameter settings. More efficient search methods could probably be developed to accomplish the minimization but it is felt that the pattern search is sufficient for this study. The next section will present two examples of this method used to design controllers for second order systems.

5. DESIGN OF SECOND-ORDER PWL CONTROLLERS

The real test of any design method for pwl switching surfaces is the actual performance of various systems when operating with the pwl controller. This section presents two examples of controller design for second-order systems. Controllers were designed for $1/s^2$ and $1/s(s+a)$ plants. The systems were digitally modeled using finite state difference equations for simulation on an IBM 360/67 high speed, general purpose digital computer. The results of the designs are presented, and a comparison with controllers designed using Smith's [2] least squares method is made.

5.1 Double Integrator, $1/s^2$, Design Example

The double integrator, $1/s^2$, plant was chosen as the first example to test the feasibility of the worst-case cost index. The initial conditions were spaced along the positive x_1 axis as described in section 4.2. The optimal response times for the set of initial conditions were obtained using the minimum-time isochrones for the plant.†

$$
\begin{aligned}
t^* &= x_2(t_o) + \sqrt{4x_1(t_o) + 2x_2^2(t_o)} && x_1 > -\tfrac{1}{2}x_2|x_2| \\
&= |x_2| && x_1 = -\tfrac{1}{2}x_2|x_2| \qquad (5.1) \\
&= -x_2(t_o) + \sqrt{-4x_1(t_o) + 2x_2^2(t_o)} && x_1 < -\tfrac{1}{2}x_2|x_2|
\end{aligned}
$$

† See Athans and Falb [3], p 714.

For the digital simulation of the system the discrete difference equations for the plant were used.

$$(5.2) \quad x_1\left[(K+1)T\right] = x_1(KT) + T\, x_2(KT) + \frac{1}{2}\, T^2\, u(KT)$$

$$x_2\left[(K+1)T\right] = x_2(KT) + T\, u(KT)$$

These difference equations were used to reduce the computer time required to generate the suboptimal response times which are necessary when evaluating the success of each exploratory and pattern move in the pattern search. Initially, fourth-order, double precision Runga-Kutta integration was used, but computer time was excessive. Using the difference equation method computer time was approximately halved. Although the difference equation solution of the plant differential equations is less precise than the Runga-Kutta integration, no loss of accuracy was observed when points describing several trajectories were compared.

Rather than arbitrarily specifying an acceptability criterion, it was decided to allow the procedure to design the best possible controller for this first example. The fixed parameter and the variable parameter defining a one-segment controller were selected and the design procedure accomplished. The results of this design are presented in figure 13.

	Initial	Final
Cost	26.09	.8433
P(1)	-1.0	-9.41
P(2)	10.0	10.0

Figure 13. One-Segment Controller

A comparison of the pwl and optimal switching surfaces is shown in figure 14.

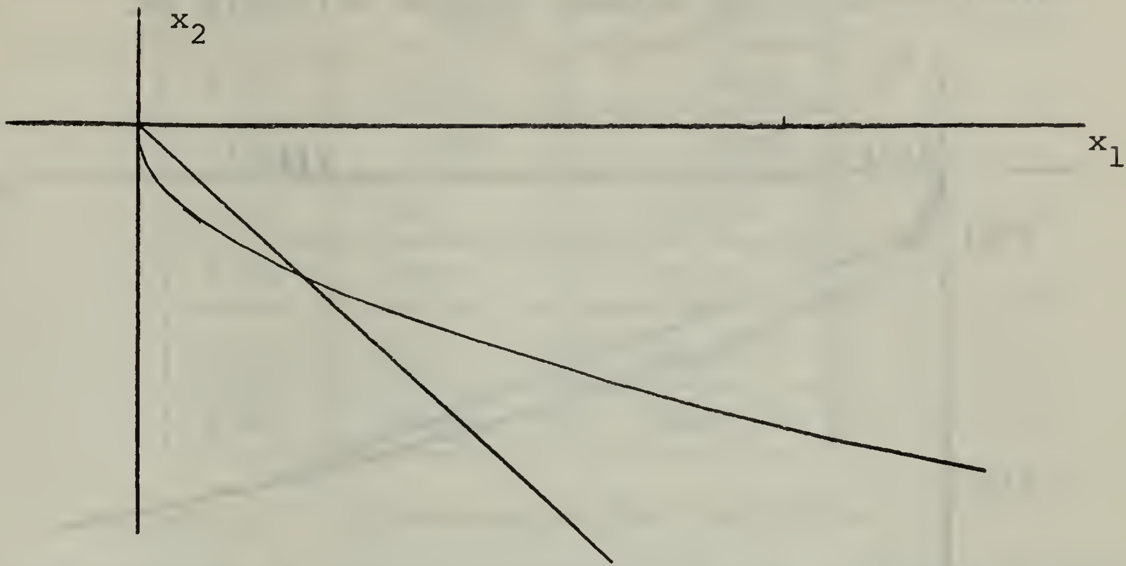


Figure 14. Optimal and Best One-Segment PWL Surfaces

The final cost listed in table I indicates that the one-segment controller causes appreciable deviation from optimal response.

In order to improve the suboptimal performance, a second pwl segment was added to the controller as called for by the design method and the increased set of parameters optimized. The results are presented in figure 15.

	Initial	Final
Cost	.8227	.0907
P(1)	-1.0	-.9812
P(2)	1.0	.4187
P(3)	-7.0	-4.572
P(4)	10.0	10.0

Figure 15. Two-Segment PWL Controller

The two-segment pwl switching surface is shown with its relationship to the optimal surface in figure 16.

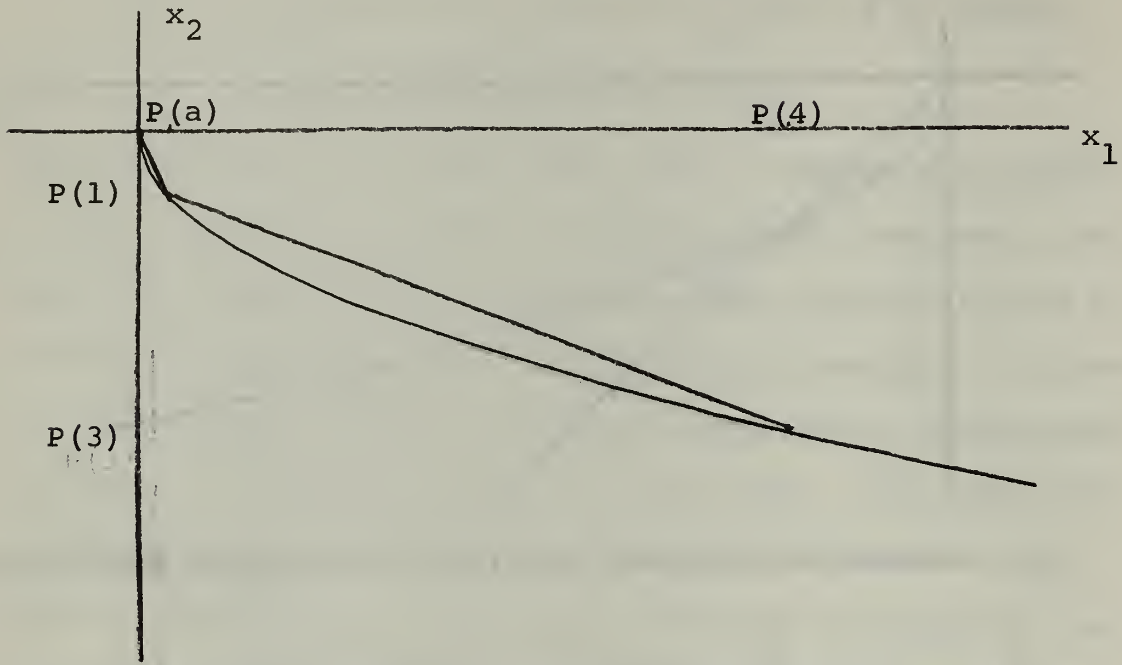


Figure 16. Optimal and Best Two-Segment PWL Surfaces

The final cost of the two-segment design given in figure 15 shows that the worst percentage deviation from optimal response is approximately nine percent. If an acceptability criterion of more than this value had been selected the pwl controller design would now be complete.

In order to determine if better pwl performance was attainable, a third segment was added to the pwl switching surface. The parameters were optimized and the results are presented in figure 17.

	Initial	Final
Cost	.2588	.0511
P(1)	-1.305	-.9456
P(2)	.744	.3868
P(3)	-2.7	-2.974
P(4)	5.0	4.4
P(5)	-5.0	-5.4
P(6)	16.0	16.0

Figure 17. Three-Segment PWL Controller.

The addition of the third segment to the switching surface decreased the cost by only .0396 while the two-segment case reduced it by .7526 over the one-segment case. Unless extremely close to optimal control is desired, the complexity added to the system by the third segment may not be justified.

This example has shown that a pwl controller can be designed for the $1/s^2$ plant that results in no more than 5.11 percent deviation from optimal response time for the set of selected initial conditions. As mentioned in section 4.2, a check must be made using other sets of initial conditions to insure that the design is acceptable over the entire range. The two-segment controller of figure 15 was evaluated with two additional sets of initial conditions in the design range. In both cases it was found that all initial conditions responded acceptably and that the parameter settings could not be improved.

The perturbation step size and initial parameter values affect the cost and final parameter values found by pattern search as noted by Hooke and Jeeves.[9] In the two segment design, costs were found to vary between .3163 and .0907 depending upon the step size. However, it was found that if the costs for different step sizes were the same, the parameter settings of the pwl surface selected by the search also agreed. This reinforces the assumption that the cost surface is a smooth function of the parameters. To insure that the best controller design results, several step sizes and initial parameter settings should be tried if the pattern search is used. Computer times using the digital model given in Appendix B averaged nine minutes for the design of the two-segment controller with the IBM 360/67 computer. The three-segment design averaged 13 minutes. Since computer time is required only during the design of the controller, these averages seem a reasonable price to pay.

5.2 $1/s(s+a)$ Design Example

The last example showed that the proposed design procedure can be used to design a pwl controller that provides acceptable performance for second order systems. As a second example, the $1/s(s+a)$ plant was chosen. This plant is better behaved than the $1/s^2$ plant since it has a pole at $s = -a$ that results in a term of the form $\exp(-at)$ in the time response. This type of response should make the plant easier to control than the $1/s^2$ plant.

The state equations for this plant are

$$\dot{x}_1(t) = x_2(t)$$

$$(5.3) \qquad \dot{x}_2(t) = -ax_2(t) + u \qquad a > 0$$

The discrete difference equations are found to be

$$x_1[(K+1)T] = x_1(KT)+(1/a-1/a\ \exp(-aT))x_2(KT)$$

$$(5.4) \qquad +(-1/a^2+T/a+\exp(-aT)/a^2)U(KT)$$

$$x_2[(K+1)T] = \exp(-aT)x_2(KT)+(1/a+\exp(-aT)/a)U(KT)$$

The initial conditions were chosen to lie along the x_1 axis with the limit at $x_1 = 15$. For this example a value of $a = 1$ was used.

The optimal response times and the set of initial conditions were generated by running the plant backward in time, switching the control once, and recording the time and x_1 coordinate when the trajectory intersected the x_1 axis. A word of caution is in order at this point if optimal response times and initial conditions are to be generated in this manner. For this plant a limit line occurs in the state space at $x_2 = -1/a$ in the fourth quadrant and at $x_2 = 1/a$ in the second quadrant. This means that trajectories starting to the right of the optimal switching surface and above the limit line will never go below $x_2 = -1/a$ before reaching the origin. Care must be taken to insure that switching of the backward time trajectories does not take place below the limit line if it is desired to generate a set of initial conditions along

the x_1 axis.

After obtaining the optimal response times and initial conditions, the design process begins by selecting an initial value for the variable parameter of a one-segment pwl switching surface. The second parameter is fixed by the range of initial conditions. For this example P(2)=16 was selected. Minimization of the cost index using the pattern search was accomplished with the results recorded in figure 18.

	Initial	Final
Cost	.2098	.00645
P(1)	-30.0	-55.2
P(2)	16.0	16.0

Figure 18. 1/s(s+a) One-Segment PWL Controller

The optimal switching surface for this plant is very nearly linear as shown in figure 19.

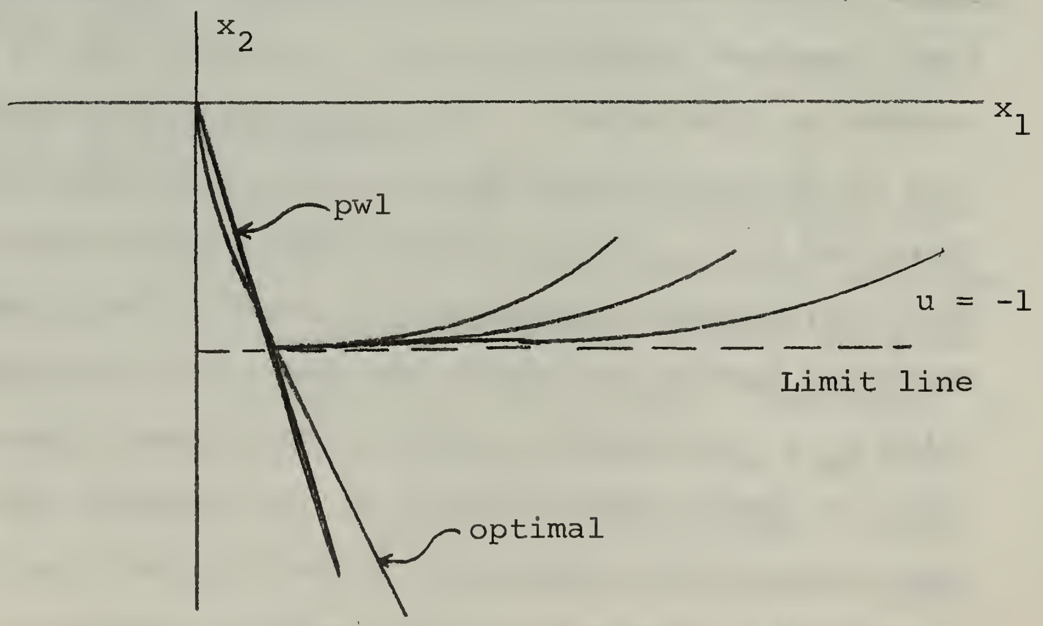


Figure 19. Optimal Switching Surface for 1/s(s+a)

The linearity of the optimal surface allows the one-segment pwl surface to provide control that is very close to the optimal. The maximum percentage deviation from optimal response time for the set of initial conditions is shown in the table to be .645%. As in the last example two other sets of initial conditions in the design range were tested with a pwl controller with the parameter settings of figure 18 and no worse deviation occurred than that found upon completion of the design.

Since trajectories for initial conditions along the x_1 axis cannot pass below the limit line, they approach the switching line as shown in figure 19. Trajectories from initial conditions beyond $x_1 = 3.8$ reach the switching line very close to its intersection with the limit line. The design procedure can minimize the cost index for any number of segments by passing the first segment through this point so that very close to optimal control results as it did in the one-segment case. A second pwl segment was added to the pwl switching surface to see if the above was true or if improvement in the design would result. The optimization procedure adjusted the first segment so that it did pass through the intersection and a cost identical to that obtained for the one-segment design resulted. If the first segment were extended the same parameter values as listed in table IV would result. After the first segment is extended beyond the intersection, the trajectories no longer intersect the second segment and no

change in the initial parameter settings is made. Since no improvement is obtained by adding the second pwl segment to the controller the design is completed with one-segment. It is to be noted, however, that this probably would not be the case for initial conditions located in other regions of the state space. The procedure could be used to design an acceptable controller for sets of initial conditions in other regions of the state space if the optimal response times were computed.

5.3 Comparison of Design Methods.

In this section the two-segment pwl controller designed in section 5.1 will be compared with controllers designed using Smith's [2] least squares, fixed-breakpoint method. For the first the x_1 breakpoints were selected by equally dividing the range of interest along the coordinate axis as specified by Smith. This resulted in one breakpoint at $x_1 = 0$ and 9.5. The least squares fit was accomplished to determine the slopes of the segments. Smith's method with equally spaced breakpoints gives the following parameter values when converted to the notation of this paper.

Method	Proposed	Smith
P(1)	-.9812	-4.67
P(2)	.4187	9.5
P(3)	-4.572	-4.917
P(4)	10.0	19.0

The pwl switching surface of section 5.1 and that designed with the least squares method are shown below.

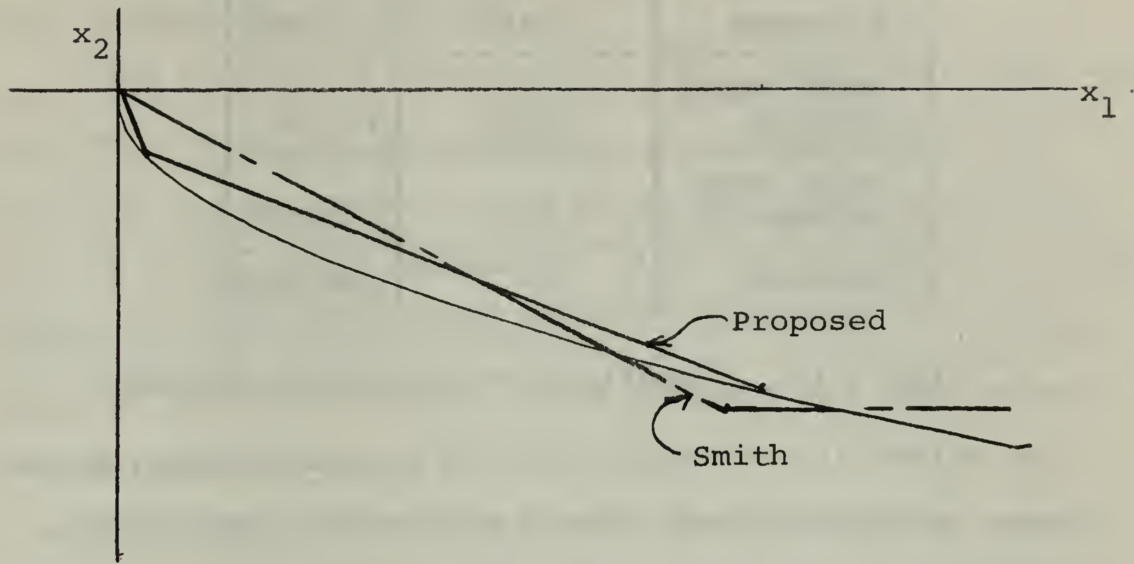


Figure 20. Comparison of Switching Surfaces.

A second controller was designed for comparison using the least squares method, but the x_1 breakpoint used was the one selected by the proposed method for the best two-segment controller given in figure 15. The parameters resulting from the fit are given below.

Method	Proposed	Smith
P(1)	-.9812	-.9959
P(2)	.4187	.4187
P(3)	-4.572	-4.1616
P(4)	10.0	10.0

The controllers were compared on the basis of maximum normalized deviation time and the sum of pwl response times for a specified set of initial conditions. The results are listed in figure 21.

Method	$\max(\hat{t}-t^*)/t^*$	Σt_i
Proposed	.092	71.460
Smith equal spacing	1.716	119.02
Smith best breakpoint	.115	73.594
Optimal	--	66.8165

Figure 21. Comparison of Controller Designs

The values in the table show the proposed design procedure is significantly better than Smith's method when evaluated using the cost index of this paper and the summation of response times.

6. SUMMARY

6.1 Discussion of Results

The problem of designing an easily-realized, close-to-optimal controller was studied. It was specified that the control law be realized as a piecewise-linear combination of the system state variables. With the controller in this form, the problem is to select the parameters that define the pwl switching surface which provides the best suboptimal performance.

The viewpoint was taken that deviations from optimal performance would be accepted if savings in terms of cost and complexity could be achieved. This approach led to the concept of acceptable performance of the controller. The properties of various cost indices were investigated, and the decision was made that a response-time index should be selected rather than a heuristic one such as the least squares polynomial fit. The problems associated with summation of response-time indices were explored, and no solution was readily apparent. The worst-case indices were then evaluated, and it was determined that if the min-max point can be found, a controller can be designed that will provide acceptable performance for the region of interest in the state space.

A design procedure that provides acceptable control while requiring the minimum number of pwl segments was proposed. Performance of the method for second-order systems

was demonstrated by example and a comparison of performance was made with Smith's[2] least squares method. The results of the comparison showed that the design method of this paper produced a significantly better controller. The design method of section 4.1 resulted in controllers that allowed only slight deviations from optimal performance. Additionally the controllers are simple to implement since they use the minimum number of pwl segments.

6.2 Areas for Further Study

Although this study has shown the merits of using the worst-case cost index, many areas of the suboptimal switching problem are open to further study. Perhaps the most important addition to this work would be its extension to third and higher order systems. As noted previously, all system simulations were accomplished using a purely digital model. The use of a hybrid model should be investigated since a considerable saving in computer time required for the design could be realized. The numerous integrations necessary to generate the pwl response times are more suitable to analog simulation while the logic capabilities of digital computation are necessary for parameter optimization.

The pattern search as used with this design method should be investigated to see if it can be modified to remove its dependence upon parameter step size and starting point of the search. In the example studies of section 5.1 and 5.2, it was determined that certain combinations

of step size and initial parameter settings caused the search to terminate prior to reaching the minimum of the cost surface. It is desired that the search reach the min-max point under all conditions. Perhaps a new search technique especially suited for finding the min-max solution regardless of the starting conditions can be developed.

The feasibility of searching the range of initial conditions in an effort to locate the true min-max should be investigated. It is possible that a point exists in the range of initial conditions that results in worse deviation than that found for the finite set used. In the examples of section 5 other sets of initial conditions in the design range were used to check for possible worse points. This check showed no worse points, however, it did not prove that none exist.

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APPENDIX A

Pattern Search for Cost Index Minimization

The pattern search technique of Hooke and Jeeves[9] is a "direct search" in that sequential trial solutions are compared to the previous "best" solution to determine the next trial solution. The search minimizes a function of several variables, S(p), by determining the sequence of values of p that provide improvement in the function value.

Two types of moves are made in the search. Exploratory moves are made about a selected base point by changing each variable by a selected step size. Each exploratory move is successful if S(p) decreases. The exploratory moves provide information about the behavior of the function being searched. The successful exploratory moves are formed into a pattern indicating the direction of a move that will probably be successful in reducing the value of the function. The sequence of exploratory moves and the establishment of a pattern is shown in figure A-1.

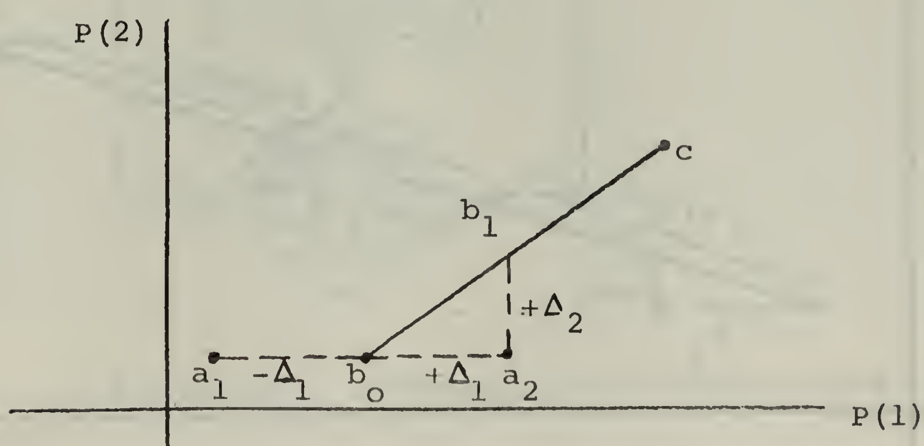

Figure A-1. Exploration and Pattern Establishment.

Starting at the first base point in the figure, point b_o, p(1) is varied by $-\Delta_1$ and S(p) is evaluated. If the value of the function is reduced the move is successful and point a_1 would become the new base point for further exploration. For the case shown, the $-\Delta_1$ move is not successful so $+\Delta_1$ is tried and found to be successful. Point a_2 now becomes the base point for variation of p(2). A step of $+\Delta_2$ is now made in p(2) and found to be successful. Point b_2 is then designated the new base point.

Rather than repeating the sequence of exploratory moves around the new base point, a pattern move is made that repeats the combined moves from the previous base point b_o. That is, all variables are again changed by the amount that was necessary to reach the present base point. In figure A-1 the pattern move is made from b_1 to c. If the move is successful, a new set of explorations is made, a new base point is found, and the pattern move is repeated. This process tends to accelerate the search as shown in figure A-2.

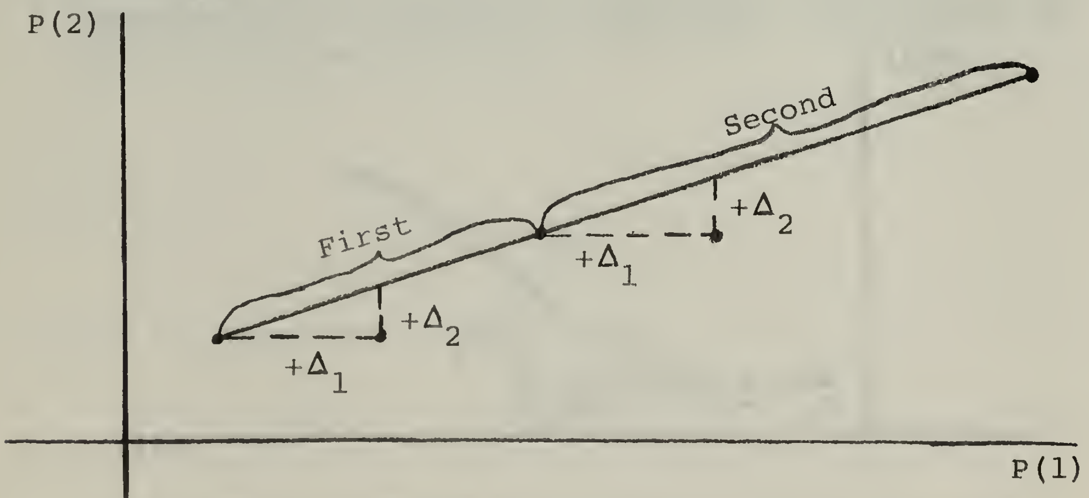

Figure A-2. Acceleration of Pattern Moves.

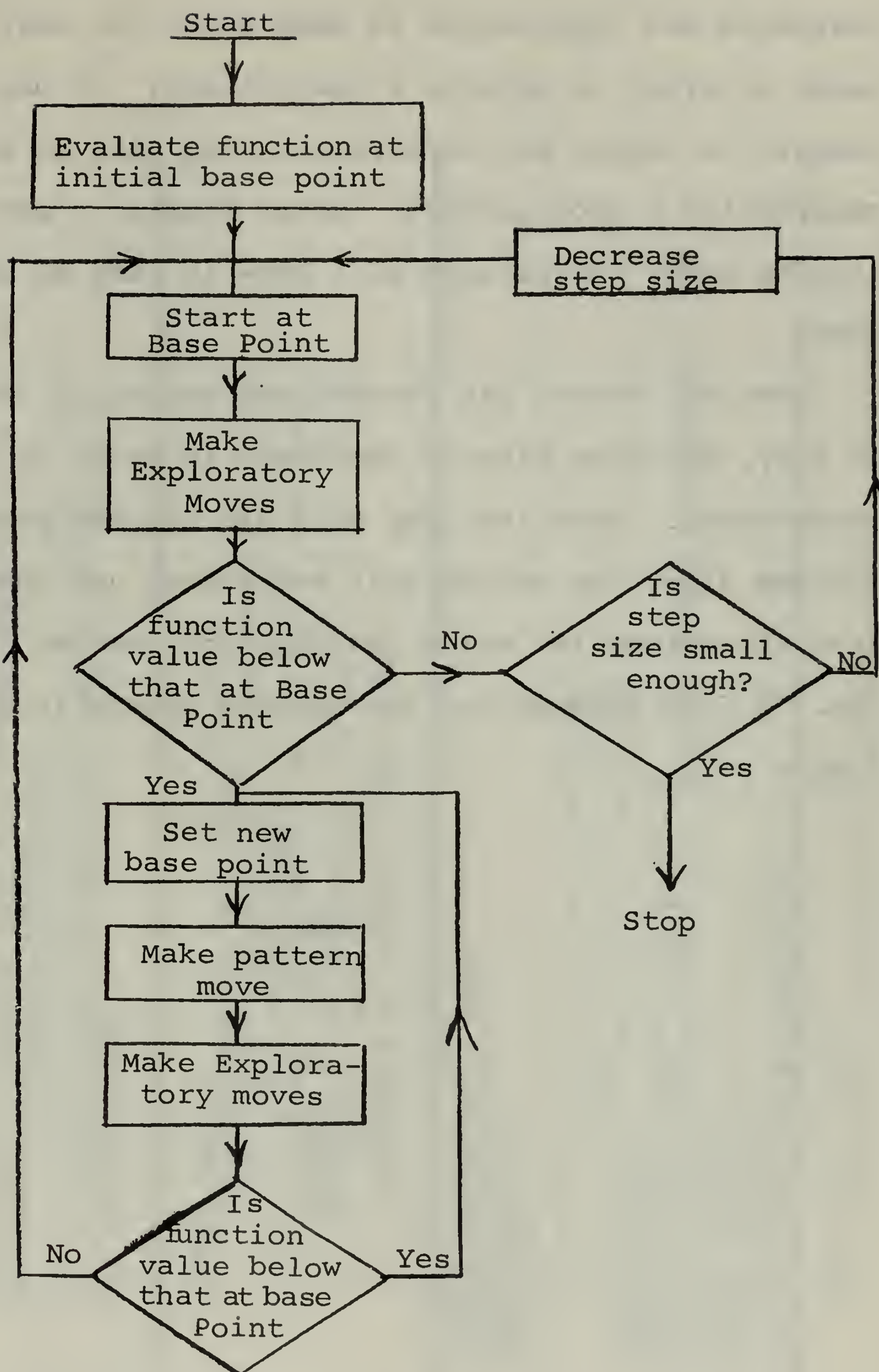

Figure A-3. Pattern Search Flow Diagram

If the pattern move is unsuccessful, the pattern is destroyed and exploration is made about the last base point in order to develop a new pattern. It may be necessary to reduce the exploration step size in order to reestablish a good pattern. After finding a new direction to move, the pattern will grow in size as described above.

When the search has reached the region of the minimum of S(p), the step size is decreased in order to continue improvement. When the step size has reached prespecified minimum size, the search will terminate, and the variables have the values for which the function reaches its minimum. A flow diagram for the pattern search is shown in figure A-3.

APPENDIX B

Computer Programs

$1/s^2$ Two-Segment Example

```
      EXTERNAL PI
      COMMON X(5),XI(5,30),XP(5),THAT(30),TSTAR(30),COST(30),DEV(30),
     1DEVN(30),PS(15),P(15),DEL,CRIT,CR,N,NSEG,NIC,NP
      DIMENSION DN(15)
C
C
C         THIS PROGRAM LOCATES THE MINIMUM IN THE COST SURFACE BY
C         MINIMIZING THE COST INDEX
C                  J=(THAT-TSTAR)/TSTAR
C
C         THE PATTERN SEARCH OF HOOKE AND JEEVES IS USED FOR
C         THE MINIMIZATION
C
C*************************************************************************
C
C         COMMON INPUT
C              DEL=INTEGRATION STEP SIZE
C              CRIT=REGION INDICATING THAT SYSTEM HAS REACHED
C                   STATE SPACE ORIGIN
C              NSEG=NUMBER OF SEGMENTS IN PWL SWITCHING SURFACE
C              N=ORDER OF SYSTEM
C              NP=NUMBER OF INDEPENDENT VARIABLES TO BE VARIED
C                  BY THE PATTERN SEARCH
C              NIC=NUMBER OF INITIAL CONDITIONS
C
C*************************************************************************
      READ(5,110)DEL,CRIT,NSEG,N,NP
  110 FORMAT(F12.8,F12.8,3I2)
    2 READ(5,103) NIC
  103 FORMAT(I2)
C
C         INPUT THE SET OF INITIAL CONDITIONS
C
      READ(5,111) ((XI(I,J) ,I=1,N),J=1,NIC)
  111 FORMAT(F12.8)
      WRITE(6,112) DEL,CRIT,NSEG,NIC,N
  112 FORMAT(2X'DEL='F12.8,5X'CRIT = 'F12.8,3X'NSEG='I2,3X'NIC= 'I2,
     13X'N = 'I2//)
  300 FORMAT(2(5XF12.8))
      DO 301 J=1,NIC
  301 WRITE(6,300) XI(1,J),XI(2,J)
C*************************************************************************
C
C
C         ADDITIONAL INPUT
C              KN PARAMETER CONTROLLING OUTPUT OF DIRECT SEARCH
```

```
C                 SUBROUTINE. SET=-1
C             CR= A PARAMETER REQUIRED IN SUBROUTINE OPT. INDICATES
C                 IF AN INITIAL CONDITION LIES ON THE OPTIMAL
C                 SWITCHING SURFACE.
C             MAXEV=MAXIMUM NUMBER OF EVALUATIONS OF FUNCTION
C                 ALLOWED BY THE PATTERN SEARCH
C             DL=INITIAL PERTURBATION STEP SIZE FOR
C                 INDEPENDENT VARIABLES.
C             DELMIN= STOPPING CRITERION FOR PATTERN SEARCH.
C             RHO=STEP REDUCTION FACTOR
C             PS(I)= VECTOR OF STARTING VALUES FOR INDEPENDENT
C                 VARIABLES
C         DN(I) IS THE VECTOR OF NORMALIZED INDEPENDENT VARIABLES.
C             THIS ALLOWS THE ACTUAL INDEPENDENT VARIABLES TO BE
C             VARIED BY DIFFERING AMOUNTS.
C
C*************************************************************************
    1 KN = -1
      CR=.0001
      MAXEV = 100
      DL=.02
      DELMIN = .02/1050.
      RHO = .25
      PS(1)=-.98
      PS(2)=.42
      PS(3)=-4.57
      PS(4)=10.
      DO 3 I = 1,NP
    3 DN(I) = 1.
C
C         GENERATE OPTIMAL RESPONSE TIMES
C
      CALL OPT
C*************************************************************************
C
C         MINIMIZE COST INDEX USING PATTERN SEARCH
C         CALLING ARGUMENTS NOT PREVIOUSLY DEFINED
C             SPI=OUTPUT VALUE REPRESENTING MINIMUM OF COST INDEX
C             PI=NAME OF FUNCTION TO BE MINIMIZED(SUPPLIED
C                 BY USER)
C             KON=OUTPUT PARAMETER INDICATING SUCCESS OF SEARCH.
C
      CALL DIRECT(DN,NP,SPI,DL,RHO,DELMIN,PI,KON,MAXEV,KN)
      IF(KON.GE.0) GO TO 40
      STOP
```

```
   40 WRITE(6,100) SPI,KON
C
C        P(I) ARE THE ACTUAL VALUES OF THE INDEPENDENT VARIABLES
C             P(I)=PS(I)*DN(I)
C
      DO 5 I=1,NP
    5 P(I) = PS(I)*DN(I)
      P(NP+1) = PS(NP+1)
  100 FORMAT(2X'JMIN = ' E18.8,5X'KON = 'I3//)
      WRITE(6,102)
  102 FORMAT(10X'OPTIMAL PARAMETER VALUES'/)
      WRITE(6,101) (P(I),I=1,NP)
  101 FORMAT(5(2XE18.8))
   50 WRITE(6,104)
  104 FORMAT(T4,'X(1)0',T22,'X(2)0',T35,'TSTAR',T52,'THAT',T65,'DEVN',T
     183,'DEV',T100,'COST'/)
  105 FORMAT(T3,F12.8,T15,F12.8,T30,E14.6,T47,E14.6,T63,E14.6,T79,E14.6
     1,T95,E14.6)
      DO 51 I = 1,NIC
   51 WRITE(6,105) XI(1,I),XI(2,I),TSTAR(I),THAT(I),DEVN(I),DEV(I),
     1COST(I)
      END
C
C
C
C
      SUBROUTINE DIRECT (PSI,K,SPSI,DELCAP,RHO,DELLC,S,KONVRG,MAXEV ,KN)
      DIMENSION PSI(15),PHI(15),SLC(15)
      INTEGER EVAL
C
      IF(K.GT.15) GO TO 50
      IF(K) 50,50,4
    4 IF(DELCAP) 50,50,5
    5 IF(RHO) 50,50,6
    6 IF(RHO.GE.1.) GO TO 50
      IF(DELLC) 50,50,7
    7 MAXEVL = MAXEV
      IF(MAXEVL) 8,8,9
    8 MAXEVL = 500
C
    9 DO 60 I=1,K
   60 SLC(I) = DELCAP
      SPSI = S(PSI)
      EVAL = 1
C
```

```
      IF(KN) 61,1,61
   61 WRITE (6,63) DELCAP,RHO,DELLC,MAXEVL,KN,(I,I=1,K)
   63 FORMAT (14H1DIRECT SEARCH,2X,8HDELCAP =,E15.6,2X,5HRHO =,E15.6,
     12X,7HDELLC =,E15.6,2X,8HMAXEVL =,I8,2X,5H KN =,I3//8H0 MOVE   ,
     215H FUNCTION VALUE,3X,3X,I2,6HST VAR,4X,3X,I2,6HND VAR,4X,
     3 3X,I2,6HRD VAR,4X,3(3X,I2,6HTH VAR,4X)/ 26X,6(3X,I2,6HTH VAR,4X)/
     4  26X,6(3X,I2,6HTH VAR,4X)   )
      WRITE ( 6,62) SPSI, (PSI(I),I=1,K)
   62 FORMAT(8H0ORIGIN ,E15.7,3X,6E15.6 /(26X,6E15.6))
C
    1 SS = SPSI
      DO 10 I=1,K
   10 PHI(I)= PSI(I)
      ASSIGN 11 TO IBK
      GO TO 40
C
   11 IF(KN) 12,13,13
   12 WRITE ( 6,14) SS,(PHI(I),I=1,K)
   14 FORMAT(8H0EXPLORE,E15.7,3X,6E15.6 /(26X,6E15.6))
C
   13 IF(SS.GE.SPSI) GO TO 3
    2 IF (EVAL.GE.MAXEVL) GO TO 51
C
      DO 20 I=1,K
      IF(SLC(I)) 21,50,22
   21 IF(PHI(I).GT.PSI(I))  SLC(I) =-SLC(I)
      GO TO 23
   22 IF(PHI(I).LT.PSI(I))  SLC(I) = -SLC(I)
   23 THET = PSI(I)
      PSI(I) = PHI(I)
   20 PHI(I) = 2.*PHI(I) - THET
C
      SPSI = SS
      SPHI=S(PHI)
      SS=SPHI
      EVAL = EVAL +1
      ASSIGN 25 TO IBK
C
   40 DO 41 I=1,K
      THET = PHI(I)
      SLCI = SLC(I)
      PHI(I) = THET + SLCI
      SPHI =S(PHI)
      EVAL = EVAL +1
      IF(SPHI.LT.SS) GO TO 42
```

```
      PHI(I) = THET - SLCI
      SPHI=S(PHI)
      EVAL = EVAL +1
      IF(SPHI.GE.SS) GO TO 44
      SLC(I)=-SLCI
   42 SS=SPHI
      GO TO 41
   44 PHI(I)=THET
   41 CONTINUE
C
      GO TO IBK,(11,25)
C
   25 IF(KN) 27,28,27
   27 WRITE( 6,29) SS,(PHI(I),I=1,K)
   29 FORMAT(8H PATTERN,E15.7,3X,6E15.6 /(26X,6E15.6))
C
   28 IF(SS.GE.SPSI) GO TO 1
      DO 26 I=1,K
      IF(ABS(PHI(I)-PSI(I)).GT.0.5*ABS(SLC(I))) GO TO 2
   26 CONTINUE
C
    3 IF(DELCAP.LT.DELLC) GO TO 52
      DELCAP = RHO * DELCAP
      DO 30 I=1,K
   30 SLC(I) = RHO * SLC(I)
      GO TO 1
C
   50 KONVRG = -1
      GO TO 53
   51 KONVRG = 0
      GO TO 53
   52 KONVRG = EVAL
   53 IF(KN) 55,54,55
   55 WRITE( 6,56)  KONVRG,SPSI,(PSI(I),I=1,K)
   56 FORMAT(10HOKONVRG = ,I10/8H EXIT    ,E15.7,3X,6E15.6/(26X,6E15.6))
   54 RETURN
      END
C
C
C
C
      FUNCTION PI(DN)
      COMMON X(5),XI(5,30),XP(5),THAT(30),TSTAR(30),COST(30),DEV(30),
     1DEVN(30),PS(15),P(15),DEL,CRIT,CR,N,NSEG,NIC,NP
      DIMENSION DN(15)
```

```
C-----THIS FUNCTION SUBPROGRAM SELECTS THE MAXIMUM VALUE OF THE
C-----NORMALIZED DEVIATION TIME FROM THE ARRAY DEVN GENERATED
C-----BY THE SUBROUTINES COST AND OPT
      DO 5 I=1,NP
    5 P(I) = PS(I)*DN(I)
      P(NP+1) = PS(NP+1)
      CALL COSMIN(DN)
      PI = 0.
      DO 20 I=1,NIC
      IF(DEVN(I).LT.PI) GO TO 20
      IMAX = I
      PI = DEVN(I)
   20 CONTINUE
      WRITE(6,700) (P(I),I=1,NP),PI,IMAX
  700 FORMAT(2(2XE15.8),2XI2)
      RETURN
      END
C
C
C
C
      SUBROUTINE COSMIN(DN)
      COMMON X(5),XI(5,30),XP(5),THAT(30),TSTAR(30),COST(30),DEV(30),
     1DEVN(30),PS(15),P(15),DEL,CRIT,CR,N,NSEG,NIC,NP
      DIMENSION DN(15)
C
C         SUBROUTINE GENERATES OPTIMAL AND PWL RESPONSE TIMES THEN
C            CALCULATES COSTS
C
C         INPUT
C         N  SYSTEM ORDER
C         NIC  NO. OF I.C.
C         DEL  INTEGRATION STEP SIZE
C         CRIT   RADIAL SIZE OF NEIGHBORHOOD SPECIFYING WHEN SYSTEM
C         IS CONSIDERED TO HAVE REACHED ORIGIN OF STATE SPACE
C         XI     MATRIX OF I.C. FOR WHICH TIMES AND COSTS DESIRED
C
C         OUTPUT
C         TSTAR VECTOR OF OPTIMAL RESPOMSE TIMES
C         THAT     VECTOR OF PWL RESPONSE TINES
C         COST=THAT/TSTAR
C         DEV=THAT-TSTAR
C         DEVN=DEV/TSTAR
C
C*****************************************************************
```

```
C
C
C         GENERATE PWL TIMES
C
      DN(4)=1.
      PS(4)=10.
      P(4)=10.
      DO 500 K=1,NP
  500 P(K)=PS(K)*DN(K)
      P(NP+1)=PS(NP+1)
      DO 502 JA=1,NIC
      RT=0.
      DO 503 L=1,N
  503 XP(L)=XI(L,JA)
C***********************************************************************
C
C         DETERMINE IF SYSTEM HAS REACHED STATE SPACE ORIGIN
C
C
C***********************************************************************
   15 IF (SQRT(XP(1)**2+XP(2)**2).LE.CRIT)GO TO 900
      U=CON(DN)
      RT=RT+DEL
      XP(1)=XP(1)+DEL*XP(2)+DEL*U*DEL/2.
      XP(2)=XP(2)+U*DEL
      GO TO 15
  900 THAT(JA)=RT
  502 CONTINUE
C***********************************************************************
C
C         COMPUTE COSTS FOR VARIOUS INDICIES
C
      DO 508 JA=1,NIC
      COST(JA)=THAT(JA)/TSTAR(JA)
      DEV(JA)=THAT(JA)-TSTAR(JA)
  508 DEVN(JA)=DEV(JA)/TSTAR(JA)
      RETURN
      END
C
C
C
C
      SUBROUTINE OPT
      COMMON X(5),XI(5,30),XP(5),THAT(30),TSTAR(30),COST(30),DEV(30),
     1DEVN(30),PS(15),P(15),DEL,CRIT,CR,N,NSEG,NIC,NP
```

```
      DIMENSION DN(15)
C
C
C        SUBROUTINE CALCULATES OPTIMAL RESPONSE TIMES FOR 1/S**2 PLANT
C        INPUT
C        X    MATRIX OF I.C. FOR WHICH OPTIMAL TIMES DESIRED
C----- CR     CRITERION TO DETERMINE IF I.C. IS ON SWITCHING SURFACE
C
C        OUTPUT
C        TSTAR  VECTOR OF OPRIMAL RESPONSE TIMES
C
      JA=1
      J=1
C
C-------------------------------------------------------------------------
C
C        THE FOLLOWING CARD MUST BE CHANGED IF ORIGIN
C        NEIGHBORHOOD IS CHANGED
C        ARB IS A FACTOR SUBTRACTED FROM THE OPTIMAL RESPONSE
C            TIME TO ACCOUNT FOR NEIGHBORHOOD OF ORIGIN.
C
C-------------------------------------------------------------------------
      ARB=.1
    4 RA=XI(2,J)*(ABS(XI(2,J)))/2
      IF(XI(2,J).LT.(-1.)*RA)GO TO 1
      IF(ABS(XI(1,J)+RA).LE.CR )GO TO 2
      TSTAR(J)=XI(2,J)+SQRT(4.*XI(1,J)+2.*XI(2,J)**2)-ARB
      GO TO 3
    1 TSTAR(J)=-XI(2,J)+SQRT(-4.*XI(1,J)+2.*XI(2,J)**2)-ARB
      GO TO 3
    2 TSTAR(J)=ABS(XI(2,J))-ARB
    3 IF(JA-1)6,7,5
    6 STOP
    7 IF(J.GE.NIC) GO TO 5
      J=J+1
      GO TO 4
    5 RETURN
      END
C
C
C
C
      FUNCTION CON(DN)
C----- THIS FUNCTION SUBPROGRAM CALCULATES THE CONTROL VALUE
C----- FROM THE STATE COORDINATE AND THE ORDINATE OF THE SWITCHING CURVE
```

```
      COMMON X(5),XI(5,30),XP(5),THAT(30),TSTAR(30),COST(30),DEV(30),
     1DEVN(30),PS(15),P(15),DEL,CRIT,CR,N,NSEG,NIC,NP
      DIMENSION DN(15)
      Y=XHAT2(DN)
      CON=1.
      IF(XP(1).GT.0.)GO TO 15
      IF(XP(2).GT.0.)GO TO 17
      RETURN
   15 IF(XP(2).LT.0.)GO TO 16
      CON=-1.
      RETURN
   16 IF(XP(2).LT.Y)GO TO 10
      CON=-1.
      RETURN
   10 CON=+1.
      RETURN
   17 IF(XP(2).LT.-Y)GO TO 10
      CON=-1.
      RETURN
      END
C
C
C
      FUNCTION XHAT2(DN)
C----- THIS SUBPROGRAM CALCULATES THE X(2) COORDINATE OF THE SWITCHING
C----- LINE FOR X(1)>0 FOR A SPECIFIED VALUE OF X(1).
C----- P(1) AND P(2) ARE THE X(2) AND X(1) COORDINATES(RESPECTIVELY)
C----- OF THE END OF THE FIRST LINE SEGMENT, P(3) AND P(4) ARE THE
C----- X(2),X(1) COORDINATES OF THE END OF THE SECOND LINE SEGMENT,ETC,
      COMMON X(5),XI(5,30),XP(5),THAT(30),TSTAR(30),COST(30),DEV(30),
     1DEVN(30),PS(15),P(15),DEL,CRIT,CR,N,NSEG,NIC,NP
      DIMENSION DN(15)
      XA=ABS(XP(1))
      IF(XA.LT.P(2))GO TO 3
      DO 1 K=1,NSEG
      KPE=2*K
      IF(XA.LT.P(KPE))GO TO 2
    1 CONTINUE
    2 XHAT2=(P(KPE-1)-P(KPE-3))/(P(KPE)-P(KPE-2))*(XA-P(KPE-2))+P(KPE-3
     1)
      RETURN
    3 XHAT2=P(1)/P(2)*XA
      RETURN
      END
```

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13. ABSTRACT

The design of controllers to provide quasi-minimum time-response is investigated. With the control law to be realized as a piecewise-linear combination of the instantaneous state values, the problem is to select the parameters which define the best suboptimal switching surface.

A worst-case response-time performance index is defined and shown to be suitable for use in the suboptimal controller design. A proposed design technique is described and used to determine controllers for second-order systems. The results of the illustrative examples are compared with controllers designed using a least squares polynomial fitting technique. The comparison shows that the proposed method designs significantly better controllers in terms of deviation from optimal response time.

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14 KEY WORDS	LINK A		LINK B		LINK C	
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